Fiscal Policy - Lecture Notes

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Chapter 1

The Basic Two-Period Model

1.1. Environment

The economy is populated by households and a government. They live for two periods, $t = 0$ and $t = 1$, and trade identical consumption goods and public bonds. Public bonds promise their holder one unit of the consumption good in the following period. There is no money in this economy. Agents trade public bonds using consumption goods.

A word on notation: each variable in the model takes a value in period zero and a value in period one, as indicated by their subscript. For example: x_0 and x_1. A process that is a function of time is called a *time series*. When a symbol omits the subscript, it refers to the entire time series vector: $x = (x_0, x_1)$.

1.1.1. The Government

The government demands $g = (g_0, g_1)$ consumption goods (*i.e.*, g_0 in period zero and g_1 in period one). To finance its purchases, it charges lump-sum taxes $\tau = (\tau_0, \tau_1)$ on households. Households cannot avoid paying taxes. The pair g and τ characterize *fiscal policy* in this model.

The government also raises revenue from selling new public bonds. In period zero, the price of one bond is q_0 units of the consumption good. Usually $q_0 < 1$: you pay less than one good in $t = 0$, to get one good in $t = 1$. As such,

$$1 + r_0 = \frac{1}{q_0}$$

is the interest rate implied by the public bond's price. In period one, agents have no incentive to save since the world ends in the following period. Since bonds have no demand, we can set its equilibrium price to zero: $q_1 = 0$.

We make two critical assumptions on government behavior. First, it can *credibly* commit to fully repaying previously issued debt. "Credibly" means that households believe in its commitment, and act accordingly. Second, the government indeed never defaults.

The government brings to period zero a debt of b_{-1} bonds, and must therefore come up with b_{-1} consumption goods to pay bondholders. To that end, it can either sell new bonds b_0 and raise $q_0 b_0$ goods in revenue, or run a *primary surplus*. The primary surplus is defined as the difference between tax proceeds and non-interest spending. In this model, it corresponds to the quantity $\tau_0 - g_0$. The government avoids a default in period zero if

$$q_0 b_0 + \tau_0 - g_0 = b_{-1}. \tag{1.1}$$

The revenue from selling new bonds plus the revenue from taxes in excess of public spending must be enough to redeem old bonds. Since the government will not default, condition (1.1) represents a budget constraint. It restricts the government's choice of how much to tax, how much to spend, and how much to borrow.

Like in period zero, in period one the government again must pay bondholders, which are now due b_0 units of the consumption good. But, in period one, the government cannot sell new bonds, since there is no demand for them (the bond price is zero $q_1 = 0$, so the government would not raise any revenues anyway). Therefore, to pay bondholders, the government must run a primary surplus of b_0 in period one:

$$\tau_1 - g_1 = b_0. \tag{1.2}$$

Expression (1.2) is also a government budget constraint.

1.1.2. Households

The consumption good is non-durable (households can only enjoy them for a single period), and perishable (agents cannot store them). Households value the consumption good in the period they make use of them. The utility function

$$u(c_0) + \beta u(c_1)$$

captures households' preferences over the amount consumed in period zero c_0 and period one c_1. Period utility $u(c)$ is an increasing, strictly concave and twice differentiable function. Parameter $\beta \in (0, 1]$ discounts the flow of future consumption, and therefore captures households' impatience.

Each household receives an endowment of $y = (y_0, y_1)$ consumption goods. You can think of households producing these goods at home; we later model firms, production and labor income more realistically.

We normalize the number of households to one, which avoids the introduction of unnecessary notation. If each household consumes c_0 goods, aggregate consumption will be

$$c_0 \times \text{Number of Households} = c_0 \times 1 = c_0.$$

The same symbol c_0 represents both individual and aggregate consumption. Likewise, (y_0, y_1) represent aggregate production in the economy.

In period zero, each household brings a_{-1} public bonds purchased in the previous period. Since households and the government are the only agents in the model, we restrict the number of bonds initially owned by households to coincide with the number of bonds owed by the government: $a_{-1} = b_{-1}$. Households redeem these a_{-1} bonds for the same number of consumption goods. Add to that their after-tax income $y_0 - \tau_0$ and we find the amount of available goods to each household in period zero. They can use these goods to consume or purchase public bonds from the government. Let a_0 be the household's choice of how many public bonds to purchase. There is no other asset in the economy, so a_0 also represents the household's savings and its net wealth. The following equation is the budget constraint faced by each household in period zero:

$$q_0 a_0 + c_0 \leq a_{-1} + y_0 - \tau_0. \tag{1.3}$$

Equation (1.3) restricts the households' decision of how much to consume and how much to save in period zero. In period one, households redeem a_0 public bonds, and do not demand new ones, as the world ends thereafter. Hence:

$$c_1 \leq a_0 + y_1 - \tau_1. \tag{1.4}$$

Households can borrow too, and the government can lend. While we have referred to b_0 as government "borrowing" and a_0 as household "savings", nothing precludes these variables from being negative (in which case, the household borrows and the government lends).

Suppose households exhaust their available resources, that is, that their budget constraints hold with equality. By equation (1.4), the maximum amount of goods a household can repay from previously acquired debt is $y_1 - \tau_1$ (in that case, the household would consume zero goods in period one, $c_1 = 0$). If the household's debt is larger than $y_1 - \tau_1$, the household defaults. Knowing that, potential lenders (other households or the government) refuse to purchase bonds from (*i.e.*, lend to) a household whose debt exceeds this value. Therefore, the largest debt any household can owe is $y_1 - \tau_1$. We incorporate this *borrowing constraint* in the model by establishing a lower bound $\underline{a}$ on period-zero savings a_0:

$$a_0 \geq \underline{a} = -(y_1 - \tau_1). \tag{1.5}$$

(If you get confused with signs, think of an example; if after-tax income equals 5 goods, then debt cannot be higher than 5, so net wealth cannot be lower than $\underline{a} = -5$.)

Economists often refer to a household's maximum repayable debt as its *natural borrowing limit*. In our model, the natural borrowing limit is $-\underline{a} = y_1 - \tau_1$. Other choices of borrowing limit $-\underline{a}$ are possible, and often more realistic. However, adopting the natural borrowing limit is a convenient starting point to analyze households' allocation decisions, because any choice that involves a positive consumption in period one ($c_1 > 0$) necessarily satisfies it. Consequently, if we prove that period-one consumption is not zero, we can safely ignore the borrowing limit.

Households decide how much to consume $c = (c_0, c_1)$ and how many bonds to purchase (or issue) a_0 taking into account their budget and borrowing constraints (1.3)-(1.5). They take the price of public bonds q_0 as given (*i.e.*, they act *competitively*), and attempt to get as much utility as possible from their choice. Therefore, the choice of how much to consume and save solves the following utility maximization problem:

$$\max_{c \geq 0, a_0} \quad u(c_0) + \beta u(c_1) \tag{1.6}$$

$$\text{s.t.} \qquad q_0 a_0 + c_0 \leq a_{-1} + y_0 - \tau_0 \tag{1.3}$$

$$c_1 \leq a_0 + y_1 - \tau_1 \tag{1.4}$$

$$a_0 \geq \underline{a}. \tag{1.5}$$

Optimization problems similar to (1.6) are often referred to as *consumption-savings* problems.

Since u is an increasing, strictly concave function, optimization (1.6) has a single solution.[1] In that solution, budget constraints (1.3) and (1.4) hold with equality - otherwise households could raise consumption and get more utility. Let $c(a_{-1}; q_0, \tau)$ and $a_0(a_{-1}; q_0, \tau)$ be the pair of consumption levels (c_0, c_1) and public bond purchases that solve (1.6). The arguments underscore how households' choices depend on their initial net wealth, the price of public bonds and taxes.

[1] We assume income y and initial wealth b_{-1} are large enough so that the household can choose non-negative amounts of consumption goods.

1.2. Present-Value Budget Constraints

1.2.1. Government and Fiscal Policy Sustainability

Let us return to the government's budget constraints, repeated below for convenience:

$$q_0 b_0 + s_0 = b_{-1} \tag{1.1}$$

$$s_1 = b_0. \tag{1.2}$$

($s = \tau - g$ is the primary surplus sequence). Equations (1.1) and (1.2) are examples of *sequential* budget constraints ("sequential" because we have one of them in each period).

Sequential budget constraints focus on the interaction between surpluses and wealth. But they also indirectly capture the possibilities of *intertemporal allocation* available to the government. For example: if it wants to lower period-zero surpluses by one ($\Delta s_0 = -1$, Δ means "a change in"), it must issue the necessary volume of new bonds $\Delta b_0 = 1/q_0 = 1 + r_0$; and then raise period-one surpluses by $\Delta s_1 = \Delta b_0 = 1/q_0$ to pay the additional debt.

It is often useful to represent the restrictions involving current and future surpluses more directly, with a single expression. Replace (1.2) on (1.1) to get:

$$b_{-1} = s_0 + q_0 s_1. \tag{1.7}$$

Equation (1.7) is the government's *present-value* budget constraint. It immediately shows that $\Delta s_0 = -1$ demands $\Delta s_1 = 1/q_0$.

We say "present-value" because we are converting spending in different points in time to their corresponding value in period zero. Indeed, the value in $t = 0$ of the delivery of X goods in $t = 1$ is $q_0 X$, since any agent can purchase X bonds for that amount, and get the X goods in $t = 1$.[1] In that sense, we can regard q_0 not only as the price of public bonds, but also the price of period-one consumption c_1 relative to period-zero consumption c_0.

We say "budget constraint" because expression (1.7) is a sufficient and necessary condition to ensure that the government does not default. Conveniently, it does not depend on the b_0 term, only on fiscal policy objects τ and g through the surplus terms $s = \tau - g$. In that sense, the present-value budget constraint implies and is implied by fiscal policy *sustainability*.

Let us check this important claim. If the government does not default, then s and

[1] This is a *no-arbitrage* argument: If the value was $A > q_0 X$, you could sell the period-one delivery of X goods for A and purchase the required bonds for $q_0 X$ to make a something-for-nothing profit.

b_0 must respect the sequential budget constraints (1.1) and (1.2). Together, they imply (1.7). Thus, no default $\implies$ the present-value budget constraint.

In the opposite direction, suppose we have a surplus process $s = (s_0, s_1)$ that satisfies (1.7). We use the period-zero sequential budget constraint (1.1) to find the necessary volume of bonds the government needs to issue:

$$b_0 = \frac{b_{-1} - s_0}{q_0}.$$

The above b_0 ensures that the government does not default in period zero. Does it default in period one? By assumption, the surplus pair satisfies (1.7). So:

$$b_{-1} = s_0 + q_0 s_1 \implies s_1 = \frac{b_{-1} - s_0}{q_0} = b_0.$$

Since $s_1 = b_0$, period-one sequential budget constraint (1.2) holds. In conclusion, validity of the present-value budget constraint $\implies$ no government default.

1.2.2. Re-Stating Households' Consumption-Savings Problem

Consider now the sequential budget constraints faced by households, expressions (1.3) and (1.4). The conclusions we find above for the government apply somewhat similarly. The sequential budget constraints imply the present-value budget constraint:

$$a_{-1} \geq [c_0 - (y_0 - \tau_0)] + q_0 [c_1 - (y_1 - \tau_1)]. \tag{1.8}$$

Each term in brackets represents the household's expenditure in excess of its after-tax income (you can think of it as the household's own "primary deficit"). The present value of its excess consumption must be lower or equal to the initial wealth a_{-1}. Intuitively, if its exceeds a_{-1}, then households default in period one.

Like in the government's case, a consumption process $c = (c_0, c_1)$ that satisfies the present-value budget constraint (1.8) also satisfies the sequential budget constraints, if we choose the right net wealth a_0. For instance, we can use period-one budget constraint, expressed with equality:

$$a_0 = c_1 - (y_1 - \tau_1). \tag{1.9}$$

The equivalency between restricting households' consumption choice using sequential or present-value budget constraints opens the door to writing the consumption-savings problem (1.6) in terms of the c only:

$$\underset{c\geq 0}{\text{Max}} \quad u(c_0) + \beta u(c_1) \tag{1.10}$$

$$\text{s.t.} \qquad a_{-1} \geq [c_0 - (y_0 - \tau_0)] + q_0 \left[c_1 - (y_1 - \tau_1)\right] \tag{1.8}$$

$$(a_0 =)\ c_1 - (y_1 - \tau_1) \geq \underline{a}. \tag{1.5}$$

(We have used (1.9) to replace a_0 in the borrowing constraint.[1]) The solution $c(a_{-1}; q_0, \tau)$ to problem (1.6) also solves problem (1.10). We can then use (1.9) again to recover the optimal demand for public bonds $a_0(a_{-1}; q_0, \tau)$.

1.3. Ricardian Equivalence

In general terms, *Ricardian equivalence* is the proposition that households' consumption choices are unaffected by the timing of taxation. In this section, we model Ricardian equivalency in our two-period setup and discuss which conditions are key to make it hold. We start with a government that fixes a fiscal policy pair g and $\tau = (\tau_0, \tau_1)$. Fiscal policy is sustainable, therefore the present-value budget constraint (1.7) is satisfied. We can write it as:

$$[\tau_0 + q_0 \tau_1] = b_{-1} + [g_0 + q_0 g_1] \,. \tag{1.11}$$

On the left, the present value of tax proceeds; on the right, the present value of outlays divided between spending and old debt redemption. Households observe the path of due taxes, and plan how much to consume $c(\tau)$ and how much to save $a_0(\tau)$.[2]

Suppose that, still at the beginning of period zero, the government announces a different, *but still sustainable*, path to lump-sum taxes, $\hat{\tau} = (\hat{\tau}_0, \hat{\tau}_1)$. Spending g remains unaltered. How do households revise their consumption plans in response to the government announcement? It turns out that, in the conditions of our two-period model, *they don't*: $c(\tau) = c(\hat{\tau})$. We say that Ricardian equivalence holds.

The key to prove the proposition is to show that different but equally sustainable taxation paths do not change the set of consumption levels affordable by households. Formally, any c that satisfies the constraints of the consumption-savings problem (1.10) under τ will continue to satisfy them under $\hat{\tau}$, and vice-versa.

Let's check that claim. We start with the present-value budget constraint (1.8),

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[1](1.9) is the only level of bond purchases consistent with a consumption choice because the sequential budget constraints hold with equality in the solution of (1.6).

[2]In this section only, I ignore the arguments a_{-1} and q_0 of the optimal solutions for brevity.

which holds with equality. We can re-write it as:

$$[c_0 + q_0 c_1] + [\tau_0 + q_0 \tau_1] - [y_0 + q_0 y_1] = a_{-1}.$$

The middle term on the left-hand side is the present value of charged taxes. Since both τ and $\hat{\tau}$ are fiscally sustainable, and since g is unchanged, that quantity must stay constant:

$$[\tau_0 + q_0 \tau_1] = [\hat{\tau}_0 + q_0 \hat{\tau}_1] = b_{-1} + [g_0 + q_0 g_1] .$$

Therefore, the household's present-value budget constraint is unchanged.

Next, consider the borrowing constraint (1.5). Since we use the natural borrowing limit, they read:

$$\begin{aligned} c_1 - (y_1 - \tau_1) = \quad & a_0 \geq \underline{a} \quad & = -(y_1 - \tau_1) \\ c_1 - (y_1 - \hat{\tau}_1) = \quad & a_0 \geq \underline{a} \quad & = -(y_1 - \hat{\tau}_1) \end{aligned}$$

Both restrictions above are satisfied whenever $c_1 \geq 0$ (this is how we define the natural borrowing limit!). Hence, the borrowing limit is effectively unchanged.

Since the restrictions of the consumption-savings problem (1.10) remain the same, the optimal level of consumption cannot be different. In conclusion, $c(\tau) = c(\hat{\tau})$.

1.3.1. Interpretation

The central idea behind Ricardian equivalence is the fact that households understand how a one-dollar reduction in charged taxes today (or a standalone one-dollar transfer) must be followed by a one-dollar increase plus interest tomorrow (and vice versa). Being the household, you can save the extra dollar, earn the interest, and duly pay the higher tax tomorrow. No reason to change the groceries list. In that sense, critics of transfer-based programs of fiscal "stimulus" often rely on the Ricardian equivalence result as a theoretical basis for their skepticism. Still, it is critical to understand what the proposition says and what it doesn't.

One could precisely summarize what Ricardian equivalence *does* say as follows:

Household's consumption demand curve does not depend on the *timing* of *lump-sum taxes*.

The two emphasized terms are key. "Timing" means *when*, not *how much*. Ricardian equivalence does not say that households do not respond to different taxation schemes. If the government halves taxes today but promises the same level of taxation in the future, households do use the additional resources to raise consumption. If the government announces higher taxes tomorrow, but no transfers today, then households save some more. (Note however that the government exhausts its resources; thus an increase in overall taxes for instance must lead to an increase in spending g too. See (1.11).) "Lump-sum" means that the proposition excludes taxes that depend on households' actions, like income, consumption and corporate taxes. Unlike these alternative forms of taxation, lump-sum taxes do not change the marginal benefits of these actions; hence, they do not induce changes in household behavior other than because they get wealthier or poorer.

1.3.2. Critical Assumptions

According to the Ricardian proposition, demand for consumption goods $c(\tau)$ is unresponsive to the timing of taxes, but not the demand for bonds $a_0(\tau)$. If the government sends you a 100-dollar check and you do not spend it, your savings account grows by 100 dollars. If the government charges you an additional 100 dollars in taxes, your savings account diminishes by that amount. One critical assumption behind Ricardian equivalence is that, if necessary, households dispose of the necessary credit to sustain their period-zero consumption level. This has been a given in our baseline case of the two-period model: under the natural borrowing limit (1.5), households can always borrow if they can repay. If the government charges 100 dollars more in taxes in $t = 0$, households can borrow an additional 100 dollars (plus interest) as lenders understand taxes will be lower by that amount in $t = 1$. The natural borrowing limit will not bind under the new path of taxes if it didn't under the old one.

However, more restrictive borrowing constraints can bind and thus prevent households from keeping their consumption path unaltered. For instance, a commonly used restriction is the *no-borrowing constraint* $\underline{a} = 0$. In our model, when the borrowing constraint binds, period-zero consumption is given by equation (1.9):

$$c_0 = a_{-1} - q_0\underline{a} + y_0 - \tau_0$$

Hence, if a fiscal policy change $\Delta\tau$ is small enough so that the borrowing constraint continues to bind, $\Delta c_0 = \Delta\tau_0$. In the presence of a binding borrowing constraint, an increase in taxation leads to a reduction in current consumption since households cannot issue more debt to pay for the higher taxes. On the opposite direction, lower taxes (or standalone transfers) might raise consumption. As such, discussions of whether adjustments to fiscal policy will stumble on Ricardian behavior often center around the extent to which households are credit constrained. Obviously, one can only answer that question empirically, on a case-by-case basis.

Also key for Ricardian equivalence to hold is the functioning of public finances, in particular the assumption that fiscal policy is credible and sustainable. In the context of real debt (*i.e.*, public bonds that pay a consumption good), fiscal sustainability is the same as no default. Our model captures best a government that is fully credible to raise enough revenue to eventually repay its debts (e.g. Switzerland). Deficits today lead to surpluses tomorrow. In practice, however, governments do default. Even if they don't, households might *believe* that they can. The credible communication of a fiscal policy plan is just as important to household behavior as the policy path itself. Whenever the government lacks the credibility of debt repayment, lower taxes today do not imply higher taxes tomorrow. Ricardian equivalence fails.

It is easy to take the assumptions of fiscal credibility and sustainability for granted, especially because most modern governments finance themselves primarily through *nominal*, not real debt. Agents redeem nominal debt for money, which is, in most cases, created by the government. Hence, unsustainable fiscal policy paths do not necessarily lead to the dramatic outcome of a government default, but rather to a decline in the value of money (inflation). We come back to that topic later. For now, just note that it is not clear how frequently and to which extent governments can and do promise fully sustainable changes in fiscal policy; and that our use of the expression "fiscal sustainability" in this section is *more restrictive* than the government not defaulting in practice.

Lastly, contrary to our model's assumptions, households are not identical, and tax and transfers are seldom unconditional. The more realistic income, capital and consumption taxes are a sure way to break Ricardian equivalence. Moreover, households with different characteristics are likely to react differently to a change in fiscal policy. We have discussed above the case of credit-constrained households. One might conjecture that older individuals will not be as inclined to save a public transfer in order to pay for a future increase in taxation. Perhaps the same applies to unemployed workers. In all, the lack of household heterogeneity is a major simplification imposed by our model.

1.4. Intertemporal Choice and Equilibrium

We want to characterize the *competitive equilibrium* of our two-period economy. The competitive equilibrium is defined by market prices and quantities that cover two properties. First, agents choose the quantities optimally, taking prices as given. The "taking prices as given" part makes the equilibrium "competitive". Second: all markets clear, which means that quantities optimally supplied equal quantities optimally demanded.

When computing an equilibrium, we fix fiscal policy (g, τ). We will later study how the government can choose fiscal policy to generate the "best" equilibrium possible. For now, we take g and τ as given, assuming that they respect the present-value budget constraint (1.7).

1.4.1. Household Optimality

Consider households' optimal choices, $c(a_{-1}; q_0, \tau)$ and $a_0(a_{-1}; q_0, \tau)$. Because they solve the consumption-savings problem (1.6) (or (1.10)), they must satisfy the first-order optimality condition associated with that problem. In an interior solution (*i.e.*, in a solution with $c_0 > 0$, $c_1 > 0$), that condition is the *Euler equation*

$$q_0 u'(c_0) = \beta u'(c_1). \tag{1.12}$$

We interpret the Euler equation (1.12) as a condition of consumption smoothing. Since the utility function u is increasing and concave, *marginal utility* u' is a positive, but *decreasing* function.[1] Intuitively, consuming more always makes the household "happier", but the amount of extra "happiness" an additional unit of consumption provides declines as it consumes more. Equating marginal utility therefore means balancing value over time. If you are lost in the desert, do not empty the waterskin on the first night.

To prove (1.12) is the first-order condition for optimality, consider the following variational argument. The utility gain of marginally increasing period-one consumption by Δc_1 is $\beta u'(c_1)\Delta c_1$. According to the present-value budget constraint (1.8), to increase period-one consumption by Δc_1, the household must give up $\Delta c_0 = -q_0 \Delta c_1$ units of period-zero consumption.

$$\begin{aligned}
a_{-1} &= [c_0 - (y_0 - \tau_0)] + q_0 [c_1 - (y_1 - \tau_1)] \\
\Delta a_{-1} &= \Delta [c_0 - (y_0 - \tau_0)] + q_0 \; \Delta [c_1 - (y_1 - \tau_1)] \\
0 &= \Delta c_0 + q_0 \Delta c_1
\end{aligned}$$

The utility loss of reducing period-zero consumption is

$$u'(c_0)\Delta c_0 = -q_0 u'(c_0)\Delta c_1.$$

For a choice of c to be optimal, the marginal gain cannot be lower or higher than the marginal loss. Thus, $q_0 u'(c_0)\Delta c_1 = \beta u'(c_1)\Delta c_1$, as we wanted to show.

The Euler equation (1.12) establishes a positive relationship between period-zero

[1] Technically, marginal utility could be zero even though utility is increasing. Here, I am assuming $u' > 0$.

and period-one consumption.

$$c_0 \uparrow \implies u'(c_0) \downarrow \implies u'(c_1) \downarrow \implies c_1 \uparrow$$

To find the actual solution $c(a_{-1}; q_0, \tau)$ to the consumption-savings problem, we impose the fact that the present-value budget constraint must hold with equality. We find the pair (c_0, c_1) that satisfies the Euler equation and that guarantees that households exhaust their available resources. Lastly, we can compute the optimal choice of period-zero savings $a_0(a_{-1}; q_0, \tau)$ using the sequential budget constraint (1.9).

1.4.2. The Competitive Equilibrium

In equilibrium, prices adjust so that markets clear. In the consumption goods market, the inelastically supplied quantity of goods y coincides with the government's demand g and households' optimal demand $c(b_{-1}; q_0, \tau)$:

$$c_0(b_{-1}; q_0, \tau) + g_0 = y_0 \tag{1.13}$$

$$c_1(b_{-1}; q_0, \tau) + g_1 = y_1. \tag{1.14}$$

(Recall $a_{-1} = b_{-1}$.) In the bonds market, the volume issued by the government coincides with that demanded by households:

$$a_0(b_{-1}; q_0, \tau) = b_0. \tag{1.15}$$

We now show that if one of these markets clears, the other two will clear as well. First, if the bonds market clears, the market for period-one consumption will also clear. Indeed, from the sequential budget constraints (1.2) and (1.4):

$$c_1 + \tau_1 - y_1 = \quad a_0 = b_0 \quad = \tau_1 - g_1.$$

The terms on the left and right imply (1.14).

Second, if the market for consumption goods clears in period zero, the market for bonds will also clear. We again see this from the sequential budget constraints (1.1) and (1.3). Subtracting the former from the latter:

$$q_0 \underbrace{(a_0 - b_0)}_{\substack{\text{Excess Demand} \\ \text{Bond Market}}} + \underbrace{c_0 + g_0 - y_0}_{\substack{\text{Excess Demand} \\ \text{Goods Market}}} = a_{-1} - b_{-1} = 0.$$

If the excess demand for goods is zero (*i.e.*, if demand = supply), the expression above implies $a_0 = b_0$.

The fact that we only need to clear one market is an application of *Walras' Law*, which states that, in an N-market economy, clearing of the first $N-1$ markets implies the clearing of the last one. Although we have three markets in our model, by now you should be convinced that the market for public bonds is really a market for period-one consumption goods. (This is the rationale behind the present-value budget constraints (1.7) and (1.8); they focus on consumption goods only).

It is convenient that we only need to clear one market, since the only price in the model is the price of public bonds q_0 (obviously this is not a coincidence). To find the equilibrium value of q_0, replace (1.13) and (1.14) in the Euler equation:

$$q_0(y,g) = \frac{1}{1+r_0(y,g)} = \beta\,\frac{u'(y_1-g_1)}{u'(y_0-g_0)}. \tag{1.16}$$

Intuitively, equilibrium bond price $q_0(y,g)$ must provide households the due incentive to allocate consumption intertemporally in a way consistent with the availability of goods. For example, suppose that period-zero endowment y_0 is much lower than period one's y_1. Under which circumstances would households accept to consume so much more in $t=1$ than in $t=0$ (so that $u'(c_1)/u'(c_0)$ is low)? According to the Euler equation: when bond prices are too low, or interest rates too high.

The equilibrium bond price (1.16) amplifies the scope of Ricardian equivalence. In the previous section, we saw that households' *demand curve* for goods are unresponsive to the timing of fiscally sustainable taxes. But demand curves are not the same as quantities demanded *in equilibrium*. In principle, the latter could change if bond prices were sensitive to taxes. Expression (1.16) proves this is not the case.

1.4.3. The Fiscal Multiplier

Given a change in public spending Δg_0, economists are often interested in the resulting change in aggregate output Δy_0. The change in aggregate output per unit of public spending $\Delta y_0/\Delta g_0$ is called the *fiscal multiplier*. In the simplified model we study in the section, aggregate output y_0 is exogenous, and unaffected by public spending. The fiscal multiplier is zero. In the following chapters we examine models that assume more elaborate production technologies and therefore allow for non-zero fiscal multipliers.

For now, a few aspects of the fiscal multiplier concept are worth noting. First, economists often limit the definition of fiscal multipliers to *exogenous* changes in public spending. "Exogenous" means that the change does not arise as a feedback response to other variables, but rather as a change in the level of spending *given* other variables.

There is no single fiscal multiplier. Even if we restrict the definition of a fiscal multiplier to encompass exogenous variations in public spending, several factors can influence their effect on the economy. Each possibility leads to a different multiplier. Here are a few examples: is the fiscal shock anticipated? Is it long-lasting? Does the government demand consumption or investment goods? We explore some of these cases in the following chapters.

Lastly, the fiscal multiplier is dual to the crowding-out effect of public spending. That is, the more output grows in response to an increase in public spending, the less private consumption needs to *decline*. You can see this from the market-clearing condition in the goods market (1.13):

$$\frac{\Delta c_0}{\Delta g_0} = \frac{\Delta y_0}{\Delta g_0} - 1$$

When the fiscal multiplier is zero, each additional good purchased by the government reduces private aggregate demand by the same amount. (In this chapter's model we only consider private consumption; we later consider private investment as well.) Based on this idea, economists sometimes claim that expansion of public spending when the economy has no spare capacity (or "slack") is detrimental to households.

Exercises

Exercise 1.1. We study the isoelastic utility function

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad \gamma > 0. \tag{1.17}$$

(a) Apply L'Hôpital's rule to show that when $\gamma \to 1$, the utility function converges to $\log(c)$.

(b) Express the Euler equation (1.12) as

$$\frac{c_1}{c_0} = [\beta(1 + r_0)]^{\gamma}.$$

The left-hand side is the gross rate of consumption growth $1 + g_1^c$. Use the first-order Taylor approximation of the log function

$$\log(1 + x) \approx x \qquad \text{when } x \approx 0$$

to conclude that

$$\gamma\left[\log\beta + r_0\right] = g_1^c.$$

The equation above show that parameter γ governs the *elasticity of intertemporal substitution*, defined by $\Delta g_1^c/\Delta r_0$.

(c) Explain intuitively why the interest rate is increasing in consumption growth.

Exercise 1.2. This exercise guides you through the complete solution of the consumption-savings problem (1.6), under the isoelastic utility function (1.17) and a general borrowing limit $\underline{a}$ (*i.e.*we no longer assume the natural borrowing limit $-(y_1 - \tau_1)$).

(a) Suppose the household has enough wealth a_{-1} to support positive consumption in period zero. Why can we guarantee positive consumption in *both* periods? Hint: consider the marginal utility of consumption as it approaches zero.

(b) Set up the Lagrangian of the optimization problem (1.6). Compute the first-order conditions to conclude that

$$q_0 u'(c_0) \geq \beta u'(c_1) \qquad (= \text{if } a_0 > \underline{a}).$$

(c) Start by assuming that the borrowing constraint $a_0 \geq \underline{a}$ does not bind. Use the Euler equation to express c_1 as a function of c_0; replace that expression on the present-value budget constraint to find solutions to c_0 and c_1, when the borrowing constraint does not bind.

(d) Replace your solution for c_0 in the period-zero sequential budget constraint (1.3) to find the required public bond position a_0. Does it satisfy the borrowing constraint? If it does, we are done. If it does not, then the borrowing solution binds.

(e) Use the sequential borrowing constraints to find c_0 and c_1 when the borrowing constraint binds.

Exercise 1.3. In this exercise we study the government's present-value budget constraint in a model with T periods.

(a) Suppose the sequential budget constraint

$$q_t b_t + s_t = b - t - 1$$

holds. Show that present-value budget constraint

$$b_{t-1} = \sum_{j=t}^{T} q_{t,j-1} s_j$$

holds, where $q_{t,j} = \prod_{i=t}^{j} q_i$. How do you interpret $q_{t,j}$? What limit condition analogous to $b_1 = 0$ in the two-period model is necessary?

(b) Show that, if the present-value budget constraint holds in every period, the sequential budget constraint holds as well (*i.e.*, the government never defaults).

Exercise 1.4. Prove Walras' Law (equilibrium in the goods market in period zero implies equilibrium in period one) using the two present-value budget constraints (1.7) and (1.8). Assume $q_0 > 0$.

Exercise 1.5. In this example, the government does not demand final goods $g = 0$ and enters period zero with no debt $b_{-1} = 0$. Households' endowment is $y_0 = 5$, $y_1 = 10$, the utility function is $u(c) = \log(c)$ and $\beta = 1$. The government transfers one consumption good to household in period zero, $\tau_0 = -1$.

(a) Find the equilibrium price of bonds and interest rate.

(b) Find the equilibrium consumption in both periods.

(c) What is the fiscally sustainable level of public transfer in period one?

(d) Compute households' savings a_0 at the end of period zero; and verify it is enough to finance their consumption and taxes in the following period.

(e) Consider a different fiscal policy. Instead of a one consumption good transfer, suppose the government enacts a one-period *tax* $\tau_0 = 1$. How do you change your answers to (a), (b), (c) and (d)?

(f) Consider now the existence of a *no-borrowing constraint*. A no-borrowing constraint is a borrowing constraint involving a zero debt limit: $-\underline{a} = 0$. That is, we change equation (1.5) to $a_0 \geq 0$. Consider again the fiscal policy change in τ you found in item (e). At the same bond price as item (a), can the household keep its consumption process unchanged? Does Ricardian equivalence hold?

(g) Under the no-borrowing constraint, is it possible to find an equilibrium with positive bond prices $q_0 > 0$ and period-zero positive taxes $\tau_0 > 0$?

Exercise 1.6. The economy is populated by a unit measure of identical households, subject to the natural borrowing limit. The government announces a new period-zero transfer of one consumption good, but only to half the population. It credibly commits to increase taxation in period one, so that the new fiscal policy remains sustainable. Based on that information, can you say that Ricardian equivalence continues to hold for sure? Can you say that it breaks? Explain.

Exercise 1.7. The government adopts a feedback rule to public spending:

$$g_0 = \theta y_0 + e_0,$$

where θ is a model parameter and e_0 is exogenously determined.

(a) Compute equilibrium output as a function of aggregate consumption c_0 and the shock e_0.

(b) Suppose $\theta > 0$. For an exogenous reason, aggregate output grows by Δc_0. Compute $\Delta y_0 / \Delta g_0$. Your favorite financial media commentator measures $\Delta y_0 / \Delta g_0 > 0$ and, based on his findings, argues that in the future the government should raise public spending in times of low output. Does the model support the commentator's claim?

Chapter 2

Production and Marginal Taxation

Capital k, depreciates at a rate $\delta > 0$. No investment cost.

Labor hours $n \in [0, 1]$. Interpret n as the share of available hours devoted to labor activity. Leisure $1 - n$.

Production function $f(k_{t-1}, n_t)$, homogeneous of degree one: $f(\alpha x) = \alpha f(x)$. Aggregate resource constraint:

$$y_t = c_t + g_t + k_t - (1-\delta)k_{t-1} \qquad t = 0, 1 \tag{2.1}$$

The term $k_t - (1-\delta)k_{t-1}$ is the aggregate investment.

Households can purchase government bonds or physical capital. There is no uncertainty, so they choose whichever offers the best after-tax return. Let d be the representative household's net wealth. Market-clearing in the capital market:

$$d_0 = q_0 b_0 + k_0. \tag{2.2}$$

Capital market closes in period one: $d_1 = b_1 = k_1 = 0$.

Capital rental rent r_t. Wage rate w_t. Marginal taxes on consumption $\tau_{c,t}$, labor income $\tau_{n,t}$ and capital income $\tau_{k,t}$. Lump-sum taxes $\tau_{L,t}$. Capital income tax applies to the net returns on both bond and physical capital investments. Depreciation deductible. The government's budget constraints are the following.

$$\begin{gathered} q_0 b_0 + \tau_{c,0} c_0 + \tau_{k,0} r_0 d_{-1} + \tau_{n,0} w_0 n_0 + \tau_{L,0} - g_0 = b_{-1} \\ \tau_{c,1} c_1 + \tau_{k,1} r_1 d_0 + \tau_{n,1} w_1 n_1 + \tau_{L,1} - g_1 = b_0 \end{gathered} \tag{2.3}$$

No-arbitrage in capital market:

$$1 + (1 - \tau_{k,1}) \left(\frac{1}{q_0} - 1 \right) = 1 + (1 - \tau_{k,1}) r_1 \tag{2.4}$$

$1/q_0 = 1 + r_1$ is the real interest rate. In the last chapter, we called r_0 the interest rate; it is now r_1 because the interest rate coincides with the cost of capital rent in period one.

Sequential representation of households' consumption-savings problem:

$$\begin{aligned} \underset{c,n,d_0}{\text{Max}} \quad & u(c_0) + v(1 - n_0) + \beta \left[u(c_1) + v(1 - n_1) \right] \\ d_0 + (1 + \tau_{c,0}) c_0 &\leq \left[1 + (1 - \tau_{k,0}) r_0 \right] d_{-1} + (1 - \tau_{n,0}) w_0 n_0 - \tau_{L,0} \\ (1 + \tau_{c,1}) c_1 &\leq \left[1 + (1 - \tau_{k,1}) r_1 \right] d_0 + (1 - \tau_{n,1}) w_1 n_1 - \tau_{L,1} \\ c_0, c_1 &\geq 0 \\ 0 \leq n_0, n_1 &\leq 1 \end{aligned} \tag{2.5}$$

First-order conditions. Euler equation:

$$\frac{u'(c_0)}{1 + \tau_{c,0}} = \beta \left[1 + (1 - \tau_{k,1}) r_1 \right] \frac{u'(c_1)}{1 + \tau_{c,1}} \tag{2.6}$$

Intratemporal condition for optimal supply of labor hours, in an interior solution:

$$v'(1 - n_t) = \frac{1 - \tau_{n,t}}{1 + \tau_{c,t}} w_t u'(c_t) \qquad t = 0, 1. \tag{2.7}$$

Marginal benefit of working one more hour = marginal cost. Setting $\tau_{n,t} = \tau_{c,t} = 0$ for brevity, consider the effect of a small change in the wage rate $\Delta w > 0$:

$$\underbrace{u'(c)\Delta w}_{\substack{\text{Substitution} \\ \text{Effect, } >0}} + \underbrace{w u''(c) \Delta c}_{\substack{\text{Wealth} \\ \text{Effect, } <0}} = -v''(1 - n) \Delta n$$

Substitution effect: supply more hours of labor because marginal benefit (wage) increases. Wealth effect: reduce supply of labor hours because higher wages leave household wealthier (alternatively: wealthier household purchases more hours of leisure). Effect of $\Delta w > 0$ on labor supply ambiguous.

Firm rents capital and hires labor hours to produce consumption goods. Firm's profit maximization problem:

$$\text{Max}_{k,n} \quad f(k,n) - (r_t + \delta)k - w_t n \qquad t = 0, 1.$$

Since f is homogeneous of degree one, maximized profit equals zero. First-order condition for optimal capital and labor demand:

$$f_k\left(\frac{k_{t-1}}{n_t}, 1\right) = r_t + \delta, \tag{2.8}$$

$$f_n\left(\frac{k_{t-1}}{n_t}, 1\right) = w_t, \qquad t = 0, 1. \tag{2.9}$$

Since f is homogeneous of degree one, derivatives f_k and f_n are homogeneous of degree zero. Both are functions *only* of the capital-labor ratio.

In equilibrium, households and firms act optimally. Labor hours and capital demanded by firms coincide with that supplied by households, in both periods. The initial conditions of the two-period economy satisfy

$$d_{-1} = q_{-1}b_{-1} + k_{-1} \qquad \text{and} \qquad 1 + r_0 = \frac{1}{q_{-1}}.$$

Therefore, Walras' Law holds. If the market for consumption goods clears in period zero, the capital market will clear; thus the market for goods in period one will clear too.

2.1. Model Implications

- Equivalency between consumption tax τ_c and labor tax τ_n.
- When labor supply is inelastic, constant consumption and labor taxes are not distortionary, like lump sum taxes.
- Capital taxation is distortionary, regardless of labor supply elasticity.
- Public debt crowds out private capital (as long as private savings not infinitely elastic).

Exercises

Exercise 2.1. In this exercise, we focus on the optimal supply of labor by the household in period zero. The properties of labor supply in period one are analogous.

(a) Mind the physical constraint on labor hours: $0 \leq n_0 \leq 1$. Set up the Lagrangean for the consumption-savings problem (2.5) to find the general first-order condition for the intratemporal choice of labor supply:

$$\begin{aligned} w_0 u'(c_0) &\geq v'(1-n_0) \qquad \text{if } n_0 > 0 \\ w_0 u'(c_0) &\leq v'(1-n_0) \qquad \text{if } n_0 < 1. \end{aligned}$$

(b) For the following items, assume $u(c)$ and $v(1-n)$ are isoelastic:

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad v(1-n) = \frac{c^{1-\frac{1}{\psi}} - 1}{1 - \frac{1}{\psi}} \tag{2.10}$$

Argue that households will not supply their entire labor endowment: $n_0 < 1$.

(c) Find the lowest level of period-zero consumption c_0 compatible with a zero supply of labor hours $n_0 = 0$. Interpret the existence of this lower bound on consumption.

(d) Show that the ψ is the *Frisch elasticity of labor supply*, defined as the change in labor hours supplied given a change in the log of the wage rate, *fixing the marginal value of consumption*:

$$\text{Frisch elasticity} = \left.\frac{\partial n_0}{\partial \log w_0}\right|_{\text{constant } u'}.$$

Hint: use the approximation $\log(1-n) = -n$ when $n \approx 0$. (Note: economists often define Frisch elasticity as the change in *log* hours, to focus on percentual change in labor hours. Here, we define it as a change in n_0 because n_0 already represents the *share* of available hours devoted to labor.)

Exercise 2.2. In this exercise, we study how the volume of taxation affects the equilibrium in the capital market, in the absence of marginal distortions. The government begins period zero with no debt $b_{-1} = 0$. Fiscal policy is characterized by a lump sum tax series $\tau = (\tau_0, \tau_1)$. There is no public spending, and no marginal taxation. Households derive no utility in leisure, and thus supply their entire endowment of working hours $n_0 = n_1 = 1$. The production function available to the representative firm displays perfect elasticity of substitution between capital and labor:

$$y_t = f(k_{t-1}, n_t) = (\bar{r} + \delta)k_{t-1} + \bar{w}n_t,$$

which implies that, in equilibrium $r_t = \bar{r}$ and $w_t = \bar{w}$, for $t = 0, 1$. Households face the natural borrowing limit.

(a) Write the household's consumption-savings problem, using the equilibrium prices. Replace its sequential budget constraints on the Euler equation to find an expression defining the optimal choice of net wealth d_0 in equilibrium.

(b) State the household's present-value budget constraint. Does Ricardian Equivalence hold?

(c) Starting from a given equilibrium, suppose the government raises lump-sum taxes in period zero τ_0 by $\Delta\tau_0 > 0$, without changing τ_1. Use the condition derived in (a) to show that $-\Delta\tau_0 < \Delta d_0 < 0$. Provide an intuition.

(d) Considering the fiscal policy change of (c), compute the change in physical capital Δk_0. What is the effect of a tax increase in period-one output?

Exercise 2.3. Suppose marginal taxes are constant: $\tau_{c,0} = \tau_{c,1}$, $\tau_{k,0} = \tau_{k,1}$. Assuming equilibrium households' consumption is also constant $c_0 = c_1 > 0$. Use the Euler equation (2.6) and the firm's capital demand schedule (2.8) to find the equilibrium interest rate r_1 and the capital labor ratio k_0/n_1. In economic models with infinite periods, these values are the *steady-state* levels of interest and capital labor ratio. Which forms of taxation affect the steady-state interest rate?

Exercise 2.4. In this exercise, we are interested in representing graphically the equilibrium in the capital markets, in a version of our two-period economy with inelastic labor supply. The functional formats are

$$u(c) = \log(c) \qquad f(k, n) = k^\alpha n^{1-\alpha}.$$

Households derive no utility in leisure, $v(\ell) = 0$, and therefore supply their entire endowment of labor: $n_t = 1$.

You should write your solution code for a general set of parameters, that you can easily change later. In the baseline specification, use $\beta = 0.75$, $\alpha = 0.5$ and $\delta = 0$. For now, we shut down the government: set all taxes, public spending and public debt to zero. The initial conditions for capital and household wealth is : $k_{-1} = d_{-1} = 1$.

Capital labor ratios k_{-1} and k_0 determine prices (w, r) through the firm's first-order conditions (2.8) and (2.9). (k_{-1} and k_0 are capital labor ratios since $n_0 = n_1 = 1$). Initial capital k_{-1} is predetermined, so we focus on k_0. Build an equally-spaced *grid* $\mathcal{K}$ for period-zero physical capital, with twenty points:

$$0.25 = \mathbf{k}_1 < \mathbf{k}_2 < \cdots < \mathbf{k}_{20} = 1.25.$$

For each $k_0 \in \mathcal{K}$, follow the steps below.

(a) Find the associated prices (w, r) using (2.8) and (2.9).

(b) Pick a one thousand-sized grid $\mathcal{D}$ of household net wealth points

$$0.25 = \mathbf{d}_1 < \mathbf{d}_2 < \cdots < \mathbf{d}_{1000} = 1.25.$$

We make $\mathcal{D}$ thinner than $\mathcal{K}$ to make sure that we approximate the optimal choice of household savings with a low error. For each wealth point $d_0 \in \mathcal{D}$, use the sequential budget constraints to find the associated period-zero and period-one consumption, loosely denoted $c_0(d_0)$ and $c_1(d_0)$.

(c) Compute households' optimal savings choice d_0^* as the $\mathcal{D}$ point that maximizes utility:

$$d_0^* = \underset{d_0 \in \mathcal{D}}{\text{Argmax}} \quad u(c_0(d_0)) + \beta u(c_1(d_0)).$$

(Whenever $c_0(d_0) < 0$, discard the candidate choice of d_0.)

(d) Repeat (a)-(c) to all $k_0 \in \mathcal{K}$. You should have a pair of vectors $r_1(k_0)$ and $d_0(k_0)$ containing the period-zero interest and households' savings for each grid point. Do higher capital points k_0 in the grid correspond to lower or higher choices of wealth d_0 by the household? Explain intuitively.

(e) Plot capital demand k_0 and capital supply $d_0(k_0)$ as functions of interest $r_1(d_0)$. Interest should be on the vertical axis of your plot.

(f) How does the equilibrium change if we make households more impatient? Repeat (a)-(d) using $\beta = 0.50$, and update your capital equilibrium plot of exercise (e) with the new capital supply curve.

Exercise 2.5. Consider again the environment of Exercise 2.4., but we now add an active government. To keep the exercise simple, the government chooses taxation parameters exogenously, and adjusts public spending to ensure fiscal policy is sustainable.

Initially, marginal taxes are fixed at 10%:

$$\tau_c = \tau_n = \tau_k = [0.1\ 0.1]',$$

and there is no lump-sum taxation, period-zero public spending is $g_0 = 0.3$. The government has no initial debt: $b_{-1} = 0$.

Your mission is to compute the equilibrium of the economy. We adopt an iterative procedure to find the equilibrium capital labor ratio, with each iteration indexed by the symbol i. Given the firms' first-order condition (2.8), searching in the space of capital labor ratios is similar to searching in the space of interest or wage rates. Start by guessing a period-zero capital labor ratio $k_0^{i=0} = 1$.

(a) Given a candidate capital labor ratio k_0^i, follow steps (a)-(c) of the previous exercise to compute households' optimal savings d_0^{*i}. Calculate the government's net debt position b_0^i in period zero, and then the stock of physical capital that clears the capital market:

$$\tilde{k}_0^i = d_0^{*i} - q_0^i b_0^i.$$

If $\tilde{k}_0^i \approx k_0^i$, stop. You have found the solution. Otherwise, you must update the capital labor ratio for the next iteration. Either set $k_0^i = \tilde{k}^i$, or use *damping* to improve numerical stability:

$$k_0^{i+1} = \sigma \tilde{k}_0^i + (1 - \sigma) k_0^i$$

where $\sigma \in (0, 1)$. After finding the equilibrium capital labor ratio, compute equilibrium r, w and c. Compute the level of government spending in period one g_1, and verify that the market for consumption goods clears.

(b) Repeat exercise (a), raising $\tau_{c,1}$ and $\tau_{k,1}$ to 0.2, one at a time. Report how wages, interest and household consumption change, and explain the new results intuitively.

Chapter 3

Income Risk and Public Insurance

3.1. Introducing Risk

Same setup as chapter 1, but now households face *idiosyncratic* income risk in period one.

$$y_1 = \begin{cases} \bar{y}_1 + z & \text{with probability } 1/2 \\ \bar{y}_1 - z & \text{with probability } 1/2 \end{cases}$$

Parameter z introduces *risk*. When $z = 0$, we recover the deterministic case $y_1 = \bar{y}_1$.

The expected value of period-one income is

$$E[y_1] = \frac{1}{2}(\bar{y}_1 + z) + \frac{1}{2}(\bar{y}_1 - z) = \bar{y}_1.$$

Given the existence of a unity measure of households, $\bar{y}_1$ is the aggregate output in period one.

We must adapt utility function to accommodate the existence of uncertainty. Assume *expected utility format*

$$u(c_0) + \beta E[u(c_1)] = u(c_0) + \beta \left[0.5\, u(c_1^H) + 0.5\, u(c_1^L)\right],$$

where c_1^H is consumption in the "high" income state, c_1^L in the "low" income state. When $z = 0$, we recover the original utility function $u(c_0) + \beta u(c_1)$. We say households are (strictly) *risk-averse* when u is (strictly) concave. We assume u to be strictly concave.

How does the introduction of risk changes demand for consumption goods and public bonds? Assume natural borrowing limit, and that $\lim_{c \to 0} u'(c) = \infty$, so that the solution to consumption is interior with respect to the borrowing limit. Consider first the original case with deterministic $y_1 = \bar{y}_1$ and no government. Let a_0^D denote public bond demand ("D" for deterministic), and the same for c^D. Optimality requires the Euler equation:

$$q_0 u'(y_0 - q_0 a_0^D) = \beta u'(a_0^D + \bar{y}_1). \tag{3.1}$$

The effect of introducing income risk on household income depends on whether u' is a concave or convex function. That is, it depends on the *third* derivative of the utility function u'''. It is common to assume $\lim_{c\to 0} u'(c) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$, suggesting that u' is a *convex* function: $u''' > 0$. This is the case with the common isoelastic utility function.

The consumption-savings problem faced by the household in the presence of income risk:

$$\begin{aligned}
&\underset{c,a_0}{\text{Max}} \quad u(c_0) + \beta \left[0.5\, u(c_1^H) + 0.5\, u(c_1^L)\right] \\
\text{s.t.} \quad & q_0 a_0 + c_0 \le y_0 && (3.2)\\
& c_1^H \le a_0 + y_1 + z && (3.3)\\
& c_1^L \le a_0 + y_1 - z && (3.4)\\
& c_0, c_1 \ge 0
\end{aligned}$$

Since $u'(0) = \infty$, households choose positive consumption in both states, and borrowing constraint does not bind. In the interior solution, the Euler equation is:

$$q_0 u'(c_0) = \beta \left[0.5\, u'(c_1^H) + 0.5\, u'(c_1^L)\right] = \beta E[u'(c_1)]. \tag{3.5}$$

Replacing the sequential budget constraints (3.2)-(3.4):

$$q_0 u'(y_0 - q_0 a_0) = \beta E[u'(a_0 + y_1)]$$

When u' is a strictly convex function ($u''' > 0$), households react to the introduction of income risk by *reducing consumption and raising demand for public bonds.* To see this, apply *Jensen's inequality* to the Euler equation:

$$\begin{aligned}
q_0 u'(y_0 - q_0 a_0) &= \beta E[u'(a_0 + y_1)] \\
&> \beta u'(a_0 + E[y_1]) \\
&= \beta u'(a_0 + \bar{y}_1)
\end{aligned}$$

Compare the inequality above with (3.1). Households react to the introduction of randomness by changing demand for bonds so as to increase period-zero marginal utility, relative to the deterministic case. How come? In the presence of risk, they equate marginal utility in $t = 0$ to *expected marginal utility of consumption* in $t = 1$ ($E[u'(c_1)]$), which is higher than the *marginal utility of expected consumption* ($u'(E[c_1])$). Intuitively, the combination of $u'(c_1^H)$ and $u'(c_1^L)$ is higher than $u'(E[c_1])$ because the value of consumption does not drop as much when c grows as it increases when c declines. Hence, to satisfy the new version of the Euler equation, households reduce consumption in period zero, and increase public bond demand a_0, a behavior called *self-insurance*. Economists also say that households engage in *precautionary savings*.

The introduction of randomness in the income process reduces household welfare, ex-ante:

$$u(c_0^D) + \beta u(c_1^D) \geq u(c_0) + \beta u(\bar{c}_1) \geq u(c_0) + \beta \left[0.5\, u(c_1^H) + 0.5\, u(c_1^L)\right]$$

(In the expression above, c represents optimal consumption in the income risk case.) The first inequality follows from optimality of c^D in the deterministic case; the second inequality follows from concavity of u (Jensen's inequality). Since we assume u to be strictly concave, the expression holds with strict inequality.

Utilitarian government can improve ex-ante welfare by charging 100% income tax in period one, and fully re-distributing proceeds.

3.2. An Environment with Elastic Labor Supply

Introduce elastic labor supply. Households remain identical in period zero, and supply their entire endowment of hours to firms: $n_0 = 1$. In period one, they value leisure, as captured by the utility function:

$$u(c_0) + \beta E\left[u(c_1) + v(1 - n_1)\right].$$

For the remainder of this section, we focus on period one. Period utility in $t = 1$ is $u(c) + v(1 - n)$, where n is number of hours devoted to labor. We assume twice differentiable, increasing, concave u and v. Additionally, $\lim_{\ell \to 0} v'(\ell) = \infty$, so households always devote some time for leisure: $n < 1$.

No physical capital. Households provide differentiated labor hours. Each household has an individual (or *idiosyncratic*) level of productivity z, meaning that n hours of its labor corresponds to $z \times n$ *efficiency hours of labor*. Efficiency hours of labor differ from physical hours of labor because they incorporate individual productivity. Random variable z can take S different values: $z_1 < z_2 < \cdots < z_S$, with probability $p_1, p_2, \ldots, p_S$, respectively. Of course, $\sum_s p_s = 1$.

Symbol	Description
n_1^z	Labor hours supply by household with productivity z
$\bar{n}_1$	Aggregate (efficiency) hours labor
$w_1 z$	Wage rate per hour of labor
w_1	Wage rate per efficiency hour of labor
$h_1 = w_1 \bar{n}_1$	Aggregate labor income
$\text{rev} = \tau h_1$	Public revenue from labor income

Table 3.1: Key Labor Market Variables in Period One

Productivity draws are independent from each other. Therefore, after draws occur, p_1 households land state $s = 1$, p_2 land $s = 2$, and so on. This is an application of the law of large numbers.

We break down production into two layers. A representative intermediary firm hires labor hours from households and builds a homogeneous "aggregate efficiency labor" commodity (or just "aggregate labor", for brevity). The representative firm that produces consumption goods uses aggregate labor as the only production factor.

The intermediary firm aggregates labor using the production function

$$\bar{n}_1 = \int_0^1 z(j)n(j)dj = p_1(z_1 n_1^{z_1}) + \cdots + p_S(z_S n_1^{z_S}) = E\left[z n_1^z\right].$$

In the integral, $z(j)$ is the productivity of household j and $n(j)$ is its labor choice. I also define n_1^z as the working hours choice made by households with productivity z. We characterize their optimal choice later.

The intermediary firm sells the $\bar{n}_1$ aggregate hours at a rate of w_1 per hour. Since technology is linear, in equilibrium the wage rate is $w_1 \times z$. Hence, we refer to w_1 as the *wage rate per efficiency hour of labor*. The aggregate labor income h_1 is

$$h_1 = \int_0^1 w_1 z(j)\tilde{n}(j)dj = w_1 \bar{n}_1.$$

Table 3.1 summarizes labor market variables in period one.

Breaking the wage rate between its idiosyncratic and common components is convenient because the number of aggregate hours of labor demanded by the consumption good producer depends only on the latter. Since we focus on taxation, suppose the final good producers converts one hour of aggregate labor into one consumption good. Hence, $w_1 = 1$. (In any case, we continue to write w_1 in the formulas, for clarity of the arguments.) In this setup, all firms are indifferent regarding production scale: their profits equal zero regardless.

3.3. Taxation and Laffer Curve

We are interested in studying if and how the government can use fiscal policy to help insure households against income risk. We start by focusing on a fiscal policy that combines a flat tax τ on labor income and lump-sum transfers R to households, both imposed only in period one. This notation simplifies the more cumbersome $\tau_{n,1}$ and $\tau_{L,1}$ symbols of chapter 2, which we can drop since there are no other taxes.

By the sequential budget constraint, period-one consumption for a household with productivity z is

$$c_1^z = a_0 + (1-\tau)w_1 z n_1^z + R.$$

The first-order condition for labor supply:

$$(1-\tau)\, w_1 z\, u'(c_1^z) \leq v'(1-n_1^z) \qquad (= \text{if } n_1^z > 0) \tag{3.6}$$

Marginal benefit of working +1 hour = marginal cost; otherwise, household are constrained. Since $v' > 0$, 100% taxation $\tau = 1$ leads to $n_1^z = 0$: households supply no hours of labor.

For the remainder of this section, we fix a_0, w_1 and R, and express optimal labor choice $n_1^z(1-\tau)$ as a function only of the net-of-tax parameter $1-\tau$. Because of substitution and wealth effects, an increase in τ has an ambiguous effect on n_1 (but we know that $n_1 = 0$ when $\tau = 1$).

With the individual labor supply $n_1^z(1-\tau)$, we can compute aggregate labor supply

$$\bar{n}_1(1-\tau) = p_1(z_1 n_1^{z_1}(1-\tau)) + \ldots p_S(z_S n_1^{z_S}(1-\tau))$$

and the aggregate labor income $h_1(1-\tau) = w_1\bar{n}_1(1-\tau)$. By charging a marginal rate τ, the government raises a total revenue of τh_1. Express that as a function of τ:

$$\text{rev}(\tau) = \tau\, h_1(1-\tau) \geq 0.$$

Function $\text{rev}(\tau)$ is known as the *Laffer curve*. Its shape depends largely on the labor supply model at hand. In general:

- $\text{rev}(0) = 0$ ($\tau = 0$, no taxes charged)
- $\text{rev}(1) = 0$ ($n_1 = \bar{n}_1 = 0$, households have no incentive to work).

In the particular case that $h' \geq 0$ and $h'' \leq 0$, the Laffer curve has an inverted-U shape. Figure 3.1 shows an example.

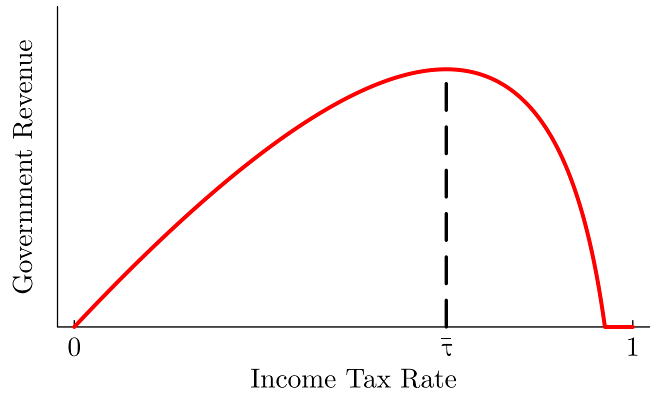


Figure 3.1: Laffer Curve Example

Revenue-maximizing tax rate $\bar{\tau}$ satisfies $\text{rev}'(\bar{\tau}) = 0$:

$$\text{rev}'(\bar{\tau}) = h_1 - \bar{\tau} h_1' = 0, \tag{3.7}$$

where h_1 and h_1' are both evaluated at the point $1 - \bar{\tau}$. Re-writing the expression above yields:

$$\bar{\tau} = \frac{1}{1+e} \qquad \text{where } e = \frac{\partial h(1-\bar{\tau})}{\partial(1-\tau)} \frac{1-\bar{\tau}}{h(1-\bar{\tau})} \tag{3.8}$$

is the elasticity of aggregate labor income to after-tax efficiency wage rate $(1-\tau)w_1$, which we can measure empirically. Higher elasticities are associated with lower optimal tax rates.

3.4. Optimal Insurance

To provide insurance against income risk, the government distributes the proceeds from the labor tax charge back to households in the form of a lump-sum transfer R. Each household receives the same transfer: we thus model a *universal basic income* program. If the government uses all available resources, and if the policy is sustainable, $R = \text{rev}(\tau)$; so we can write $R(\tau)$.

We continue to leave a_0 and w_1 fixed. Which tax rate τ maximizes household welfare ex-ante (*i.e.*, prior to the productivity draw)? Utility at the beginning of period one:

$$E\left[u(a_0 + (1-\tau)w_1 z n_1^z + R(\tau)) + v(1 - n_1^z).\right]$$

(Labor supply n_1 evaluated at $(1-\tau)$.) To facilitate notation, let

$$u'_z(\tau) = u'\left(a_0 + (1-\tau)w_1 z n_1^z(1-\tau) + R(\tau)\right)$$

be the period-one marginal utility of a household that draws z. The first-order condition for optimal tax rate τ^* is

$$E\left[u'_z(\tau^*)\right] R'(\tau^*) = E\left[u'_z(\tau^*) z n_1^z(1-\tau^*)\right] > 0.$$

(Note the application of the envelope theorem.) Since $u' > 0$, $R'(\tau^*) = \text{rev}'(\tau^*) > 0$. Compare this condition to (3.7). Isolating optimal taxation:

$$\tau^* = \frac{\lambda}{\lambda + e}, \tag{3.9}$$

where

$$\lambda = -\frac{\text{cov}\left(u'_z, z n_1^z\right)}{\bar{n}_1 E(u'_z)} \geq 0$$

measures the degree of consumption inequality after public insurance has been implemented. If the government manages to equalize consumption across households, u' is constant and hence $\lambda = 0$. Otherwise, $\lambda > 0$ since marginal utility decreases in consumption and hence in realized labor income $z n_1^z$.

Optimal taxation is increasing in inequality, and decreasing in labor supply elasticity.

Exercises

Exercise 3.1. In the context of the model with elastic labor supply and heterogeneous productivity, express aggregate labor income as a function of the labor tax rate $H_1(\tau) \equiv h_1(1-\tau)$. Show that the tax rate $\bar{\tau}$ that maximizes government revenue attains

$$-\frac{\partial H_1(\bar{\tau})}{\partial \tau} \frac{\bar{\tau}}{H_1(\bar{\tau})} = 1.$$

That is, the elasticity of H_1 with respect to the tax rate is equal to one in the revenue-maximizing point. Provide an interpretation.

Exercise 3.2. Consider a model with discrete labor choice and no uncertainty. Households can either supply their whole endowment of hours $n_1 = 1$, or no hours at all $n_1 = 0$. They enter period one with a public bonds, and are offered a wage rate of w.

Period-one utility is

$$u(c) - \zeta \, \mathbf{1}_{n=1},$$

where $\zeta > 0$, and $\mathbf{1}_{n=1}$ is an indicator function, which equals one when $n = 1$, and zero otherwise. Function u is increasing.

(a) Start by assuming there is no taxation. Compute the household's labor supply decision rule, as a function of a and w.

(b) Suppose the government charges a flat marginal tax rate $\tau_{n,1}$ on labor income, and uses the revenue for public spending (which households do not value). Re-compute the household's decision rule. Find the expression defining the threshold taxation level $\tau^*_{n,1}$ above which households opt not to work.

(c) Sketch the plot of the Laffer curve.

(d) Suppose $u(c) = \log(c)$. How does $\tau^*_{n,1}$ depend on households' wealth a? Explain intuitively.

Exercise 3.3. We consider a particular case of GHH preferences (following Greenwood, Hercowitz and Huffman). In period one, the utility function is $u(c + \Phi v(1-n))$, where u is increasing and differentiable, and

$$v(1-n) = \frac{(1-n)^{1-1/\psi}}{1-1/\psi}.$$

(a) Compute the first-order condition for the optimal choice of labor in period one. Since $v'(\ell) \to \infty$ when $\ell \to 0$, $n_1 < 1$: you only need worry about the lower bound on labor choice. Show that when $w \leq \Phi$, the household does not work: $n_1 = 0$.

(b) Use the first-order condition to argue that there is no wealth effect on labor supply.

(c) With the help of a computer, set $\Phi = \psi = 0.5$ to reproduce the Laffer curve in figure 3.1.

Exercise 3.4. *Universal basic income* (UBI) programs propose that every individual receives an unconditional transfer of money, regardless of their earnings and other aspects of tax legislation. Consider a UBI scheme that transfers R consumption goods, and taxes all households at a flat rate of τ. Show that this UBI program is economically equivalent to a non-UBI, nonlinear taxation scheme that establishes two income brackets: $h \leq \hat{h} = R/\tau$ and $h \geq \hat{h}$. Find the required tax functions $T_0(y)$ and $T_1(y)$ in each income bracket (households that earn less than $h < \hat{h}$ pay $T_0(h)$ in taxes; those who earn $h \geq \hat{h}$ pay $T_0(\hat{h}) + T_1(h - \hat{h})$).

Exercise 3.5. Consider the elastic labor supply model of the main text, in which the government inherits no public debt: $b_{-1} = 0$. All households have access to half a unit of the consumption good in period zero ($y_0 = 0.5$) from labor endowment. Households have the utility function

$$u(c_0) + \beta E\left[u(c_1) + v(1 - n_1)\right]$$

where $u(x) = v(x) = log(x)$, and $\beta = 0.8$. Productivity z can take two values: $z_1 = 1+\sigma$ and $z_2 = 1 - \sigma$, each with probability $p = 0.5$.

(a) Initially, the government does not tax or transfer goods. In equilibrium, what is the net wealth a_0 of each household in the beginning of period one? Show that the optimal labor supply is $n_1 = 0.5$, regardless of z. Compute household income as a function of productivity. (To solve this problem, recall that $w_1 = 1$.)

(b) With the help of a computer, plot the equilibrium interest rate as a function of σ (vary σ from 0 to 0.4), and provide an interpretation for your findings.

(c) Let $\sigma = 0.2$. Suppose now that the government introduces a basic income program, funded by a $\tau = 0.2\%$ flat labor income tax. Derive analytically each households' optimal labor supply n_1^z, *given* the government's transfer R. Your first task is to compute the government revenue R from taxing labor income, which depends on household labor supply (which, in turn, depends on R itself).

Write in your code a function that computes optimal labor supply given a lump-sum transfer R. Write up a second function $g(R)$ that uses the first one to calculate the government revenue from taxing households. The equilibrium revenue raised by the government by taxing labor income satisfies the fixed-point problem: $R = g(R)$. Compute R. (Tip: adopt an iterative procedure. Guess some R_0; then update your guess using $R_i = g(R_{i-1})$ until R_i is close enough to the fixed point.)

(d) With the equilibrium lump-sum transfer R, compute the equilibrium interest rate, and compare it with the interest rate arising in the absence of taxation. Explain intuitively why results differ. Does public insurance against income risk guarantee a decline in period-zero demand for bonds?

Exercise 3.6. Model with capital.

Chapter 4

Introduction to Finite-Horizon Dynamic Programming

4.1. Dynamic Programming Concepts

4.2. Adding Uncertainty

4.3. Computing Optimal Supply of Labor

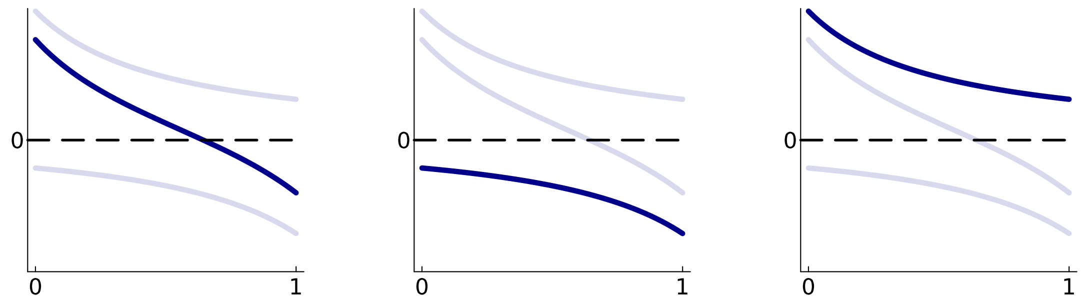
This section provides an algorithm to compute household's optimal supply of labor hours. Define the "net" marginal benefit of increasing working hours

$$h(n) = wu'(wn + z) - v'(1 - n),$$

where w is the after-tax income and a is a term that groups other components of the budget constraint, like bond redemptions, new bond purchases and government transfers. Here, we fix both w and z. When $h(n) > 0$, the marginal benefit (in utility units) of working a little more $wu'(wn + z)$ outweighs the marginal cost $v'(1 - n)$ of reducing leisure hours.

We usually assume u and v are concave, which implies that $u'', v'' < 0$ and, thus, $h'(n) < 0$. As you work more, the benefit of increasing labor hours declines - first because leisure becomes scarcer (thus more valuable, v' term) and, second, because consumption grows (thus becomes less valuable, u' term).

Let n^* be the optimal supply of labor hours. We can split the first-order condition for n^* to be optimal in three cases. Case 1: If n^* is an interior solution for the household problem, then $h(n^*) = 0$. Case 2: If $n^* = 0$ and the household is constrained by the fact



(a) Interior Solution **(b)** Lower Bound Solution **(c)** Upper Bound Solution

Figure 4.1: Marginal Net Benefit Function h: Solution Cases

that it cannot work less than zero hours, then $h(n^* = 0) \leq 0$. Case 3: If $n^* = 1$ and the household is constrained by the fact that it cannot work more than all available time, then $h(n^* = 1) \geq 0$. Figure 4.1 depicts three examples of h, each with a solution belonging to a different case.

In practice, we do not know from the beginning which case is right. However, since net marginal benefit always declines in labor hours ($h' < 0$), we know that $h'(n^*) = 0$ can only hold for a single point. We can therefore adopt the following algorithm to numerically (or analytically) compute n^*:

1. If $h(0) \leq 0$, then $n^* = 0$. Stop.
2. If $h(1) \geq 1$, then $n^* = 1$. Stop.
3. Otherwise, search for the zero of h in the interval $(0, 1)$.

If you get to the last step, then you know that $h(0) > 0$ and $h(1) < 0$ (otherwise the algorithm stops in one of the previous steps). In that case, you need to find the zero of function h, that is, the point n^* between zero and one such that $h(n^*) = 0$.

A simple *bisection method* can be applied to find the zero of h. Starting with $n_0 = 0$ and $n_1 = 1$, follow the steps below.

1. Define $n = \frac{n_0 + n_1}{2}$.
2. If $n_0 \approx n_1$ or $h(n) \approx 0$, stop. You have found the zero of h.
3. If $h(n) > 0$, set $n_0 = n$ and go back to step 1.
4. If $h(n) < 0$, set $n_1 = n$ and go back to step 1.

(The bisection method above assumes h is decreasing; if you are interested in finding the zero of an increasing function f, you can imply the steps to $-f$.)

Exercises

Exercise 4.1. Given a decreasing function f, and two points $a < b$, write the code of a function that applies the bisection method described in section 4.3 to find the zero of f between a and b.

(Tip: In the context of iterative procedures that depend on control clauses to end - like the bisection method -, it is good practice to limit the number of iterations the algorithm can perform. Otherwise, typos or unfortunate examples can lead your computer to loop over the iteration endlessly.)

Exercise 4.2. Let

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad \text{and} \qquad v(\ell) = \frac{\ell^{1-\frac{1}{\psi}} - 1}{1 - \frac{1}{\psi}}.$$

When $\gamma = 1$, $u = \log$, and the same is true for ψ and v. Write a code that applies the algorithm described in section 4.3 of this chapter to compute the optimal labor supply choice in the problem

$$\underset{n}{\text{Max}} \quad u(wn + a) + v(1 - n) \qquad \text{s.t.} \quad 0 \leq n \leq 1.$$

Use can use the bisection function you wrote in the previous exercise.

Chapter 5

Overlapping Generations and Pension Systems

5.1. OLG in Infinite Periods

Infinite periods $t = 0, 1, 2, \ldots$. Each period, new generation of households born (unity measure). Households live for two periods, "young" and "senior". Single consumption good. Young households receive an endowment of one unit of the consumption good. No government action.

Let c_s^t be period-s consumption of household born in period t, with $s \in \{t, t+1\}$. Let a_t^t be bond position (households allowed to sell bonds to each other). Linear preferences, no discounting:

$$\max_{c^t \geq 0, a_t^t} \quad c_t^t + c_{t+1}^t \tag{5.1}$$

$$\begin{aligned} \text{s.t.} \quad & q_t a_t^t + c_t^t \leq 1 \\ & c_{t+1}^t \leq a_t^t \end{aligned} \tag{5.2}$$

Finite demand for public bonds only when $q_t = 1$.

Market clearing conditions in period t:

$$\begin{aligned} c_t^t + c_t^{t-1} &= 1 \\ a_t^t &= 0 \end{aligned}$$

In equilibrium, $q_t = 1$ and each household consumes its own endowment when young.

However, this equilibrium is not Pareto optimal. *Problem of infinity.* Alternative allocation: generation born in t transfers its endowment to generation born in $t-1$. All generations left with the same single consumption good, *except $t = 0$ generation, which gets two consumption goods.*

5.2. Three-Period Environment

Three periods: $t = 0, 1, 2$. Two generations: A and B. Each with the same size of one. Generation A lives in periods zero and one, not in period two. Generation B is born in period one, and lives in period two. First period of life: "young". Second period: "senior".

Single consumption good. No capital. Households can only work when young. When senior, they receive an exogenous endowment of e units of the consumption good (home production). Linear production function $f(n) = n$ implies wage rate $w = 1$.

We initially ignore the government. Natural debt limit. Households subject to the natural debt limit. Those of generation A solve the problem

$$\begin{aligned} \underset{c^A \geq 0, a_0^A}{\text{Max}} \quad & u(c_0^A) + v(1 - n_0^A) + \beta u(c_1^A) \\ \text{s.t.} \quad q_0 a_0^A + c_0^A &\leq n_0^A \\ c_1^A &\leq a_0^A + e. \end{aligned} \tag{5.3}$$

Households of generation B solve a consumption-savings problem analogous to (5.3).

The market-clearing conditions are the following:

$$c_0^A = n_0^A \tag{5.4}$$

$$c_1^A + c_1^B = n_1^A + e \tag{5.5}$$

$$c_2^B = e \tag{5.6}$$

In equilibrium, neither generation saves or borrows - bond prices must be such that not trading in the bond market is their optimal choice.

Household heterogeneity embedded in models with overlapping generations provides an easy way to break Ricardian equivalence. The timing of taxes affects individual and aggregate demand because it affects the total income of different households.

5.3. A Pension System Model

Model with a "pay-as-you-go" pension system. Young generation B pays for senior generation A households in period one. Households from generation A face a probability $\rho \in [0, 1]$ of "retiring" in period one. We can use ρ to capture the size of the pension system as well as the retirement age.

Retired seniors receive a lump-sum transfer of ϕ consumption goods. Young households from generation B finance retirement payments through a lump-sum tax τ (we drop subscripts from τ to keep notation light - there are no other taxes). The government runs a balanced budget:

$$\rho\phi = \tau.$$

Generation A utility and consumption-savings problem:

$$\begin{aligned} \underset{c^A \geq 0, a_0^A}{\text{Max}} \quad & u(c_0^A) + v(1 - n_0^A) + \beta \left[\rho u(\tilde{c}_1^A) + (1 - \rho) u(c_1^A)\right] \\ \text{s.t.} \quad & q_0 a_0^A + c_0^A \leq n_0^A \\ & c_1^A \leq a_0^A + e \\ & \tilde{c}_1^A \leq a_0^A + e + \phi \end{aligned} \tag{5.7}$$

$\tilde{c}_1^A$ represents consumption *if the household retires*. In that case, it receives pension payment. Else, it only consumes its own savings and exogenous endowment. Utility function has *expected utility format*.

Generation B faces conventional consumption savings-problem:

$$\begin{aligned} \underset{c^B \geq 0, a_1^B}{\text{Max}} \quad & u(c_1^B) + v(1 - n_1^B) + \beta u(c_2^B) \\ \text{s.t.} \quad & q_1 a_1^B + c_1^B \leq n_1^B - \tau_{L,1} \\ & c_2^B \leq a_1^B + e \end{aligned} \tag{5.8}$$

Market-clearing conditions (5.4)-(5.6) stay the same.

Example: no leisure value $v = 0$. Therefore: $n_0^A = n_1^B = 1$. Euler equations:

$$\begin{aligned} q_0 u'(1) &= \beta \left[\rho u'(e + \phi) + (1 - \rho) u'(e)\right] \\ q_1 u'(1 - \rho\phi) &= \beta u'(e) \end{aligned}$$

Expansion of the pension system (higher ρ or higher ϕ) reduces the demand for public bonds from households in both generations, as they are left relatively richer when they are older. In equilibrium, bond prices decline, interest rates increase.

Exercises

Exercise 5.1. Consider the basic overlapping-generations model with no government. Continue to assume the linear production function $f(n) = n$, and unity wage rate. Assume $u(c) = v(c) = \log(c)$.

(a) Given β and e, find equilibrium consumption levels and bond prices.

(b) Suppose the government imposes a lump-sum tax of τ_1 consumption goods to households of generation A in period one, and τ_2 to generation B in period two. Both τ_1 and τ_2 can be negative, in which case the government is transferring goods instead taxing them. Assuming the government enters period zero with no debt, write down its sequential and present-value budget constraints.

(c) Assume the government transfers $-\tau_1 > 0$ goods to generation A households. Solve (a) under the new fiscal policy. Provide an intuition as to why the allocation and price vectors differ. Does Ricardian Equivalence hold?

Exercise 5.2. In the context of the two-period unfunded pension system model, consider again the case in which households don't value leisure, $n = 0$. Suppose the government has decided on the size τ of the pension system, but not on parameters ρ and ϕ. You can think that the government is choosing between different eligibility criteria unrelated to economic factors.

(a) Parameters ρ and ϕ must satisfy $\rho\phi = \tau$. How does the choice of ρ affect demand for public bonds by generation B households and interest rate in period one?

(b) How does it affect the demand for public bonds by generation A households and interest rate in period zero? You may assume that u' is a strictly convex function; it satisfies:

$$u'(b) > u'(a) + u''(a) \times (b - a)$$

for $a, b > 0$. Provide an interpretation based on precautionary behavior, as studied in chapter 3. (Hint: what happens when $\rho = 1$?)

Exercise 5.3. In this numerical exercise, we numerically solve the general equilibrium effects of the introduction of a realistic unfunded pension system. Following the setup of chapter 2, firms produce consumption goods using labor and physical capital through the production function

$$f(k, n) = k^{\alpha} n^{1-\alpha}.$$

Capital depreciates at a rate δ, and the expressions

$$\begin{aligned} r_t + \delta &= \alpha(k_{t-1}/n_t)^{\alpha-1} \\ w_t &= (1-\alpha)(k_{t-1}/n_t)^{\alpha} \end{aligned} \tag{5.9}$$

provide first-order conditions for optimality when firms do not profit. Since households do not work in period two, $r_2 = -\delta$. There is no senior age endowment e.

As in the main text, senior households of generation A have a probability ρ of receiving a pension installment of ϕ consumption goods. The installment is financed by a flat, marginal tax τ on labor income (different from the lump-sum tax of the main text). The government initially has no public debt, and adopts a balanced budget in all periods, which requires $\rho\phi = \tau n_1^B$.

Using the end-of-period notation, generation A households enter period zero with a net wealth of $d_{-1} = k_{-1}$. The market-clearing condition in the capital market is

$$d_t = k_t \qquad t = 0, 1. \tag{5.10}$$

Utility of generation A is similar to that of the text

$$u(c_0^A) + v(1 - n_0^A) + \beta E\left[u(c_1^A)\right],$$

with isoelastic u and v:

$$u(c) = \frac{c^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}} \qquad \text{and} \qquad v(\ell) = \frac{\ell^{1-\frac{1}{\psi}} - 1}{1 - \frac{1}{\psi}}.$$

The utility function of generation B is analogous. For the baseline calibration, use $\alpha = 0.5$, $\beta = 0.8$, $\delta = 0.1$, $\gamma = 0.5$, $\psi = 0.8$, $\rho = 0.5$ and $\phi = 0.1$. The initial stock of physical capital is $k_{-1} = 1$.

(a) Write the consumption-savings problem faced by households of generation A. Consider a grid $\mathcal{D}$ of household net wealth points:

$$0.05 = \mathbf{d}_1 < \mathbf{d}_2 < \cdots < \mathbf{d}_{1000} = 0.6$$

Write a function that takes as given w_0, r_0, r_1, ρ and ϕ, and returns the optimal choice of d_0, c_0^A, $\tilde{c}_1^A$, c_1^A and n_0^A, by households of generation A.

To solve the problem, you need to compute the utility of selecting each candidate net wealth position $d \in \mathcal{D}$, and then choose the point that maximizes it. Hint: to compute the optimal labor supply choice associated with a point d, use the function you wrote in Exercise 4.2. of chapter 4. It needs to solve

$$w_0 u'(w_0 n_0^A + (1 + r_0)d_{-1} - d_0) = v'(1 - n_0^A).$$

(b) Write the consumption-savings problem faced by households of generation B. Using the same grid $\mathcal{D}$, write another function, that takes as given w_1, r_1 and τ, and returns the optimal choice of d_1, c_1^B, c_2^B and n_1^B by households of generation B. The algorithm should be similar to the one you wrote in (a).

(c) You have solved households' consumption-savings problems. Now, you need to find market-clearing prices. By (5.9), wage and interest rates depend only on the capital-labor ratio $kn_t = k_{t-1}/n_t$. It is therefore easier to search for two market-clearing capital-labor ratios kn_0 and kn_1, then the four prices w_0, r_0, w_1, r_1. You also need to ensure that the pension system is budget-balanced through proper selection of the tax rate τ. We group these variables in a single *solution vector* x:

$$x = \begin{bmatrix} kn_0 \\ kn_1 \\ \tau \end{bmatrix}.$$

Adopt an iterative procedure. Start by guessing a solution vector $x^0 = [1,\, 1,\, 0]$. In iteration i, fix x^i and solve households' problems using (a) and (b). Use the market-clearing condition (5.10) along with optimal labor supply to compute resulting capital-labor ratios:

$$\hat{kn}_0 = \frac{k_{-1}}{n_0^{Ai}}$$
$$\hat{kn}_1 = \frac{d_0^i}{n_1^{Bi}} = \frac{k_0^i}{n_1^{Bi}}$$

(the i superscript indicates the iteration). Then, use the balanced-budget condition to find fiscally sustainable pension benefits $\tau = \rho\phi^i/n_1^{Bi}$.

Define $\hat{x} = [\hat{kn}_0\, \hat{kn}_1\, \hat{\tau}]$. If $\hat{x} = x^i$, stop. You have found the solution vector, with equilibrium capital-labor ratios and pension benefits. Otherwise, update the candidate solution vector using damping

$$x^{i+1} = 0.5 \times \hat{x} + 0.5 \times x^i.$$

and move to the next iteration. (Remember to include a maximum number of iterations in your code to avoid an endless loop of the algorithm.)

(d) Repeat your numerical computation, but shut down the pension system: $\phi = 0$. Your solution vector should yield $\tau = 0$. How does shutting down the pension system affect equilibrium aggregate consumption, stock of capital, interest and wages? Explain intuitively.

Chapter 6

Classical Theories of Monetary-Fiscal Interaction

6.1. Public Finances in the Presence of Currency

We change the nature of public bonds. Up until now, one public bond gave its holder the right to one consumption good upon maturity. We call these real bonds. In this chapter, we study fiscal policy in the presence of *nominal bonds*. Upon maturity, nominal bonds redeem for one unit of *currency*. Currency (or *money*, or *cash*) is a commodity that only the government can produce, at no cost. The *price level* P_t is the price of a consumption good in units of currency.

We use capital letters to denote nominal variables, and lowercase to denote real variables: B_t denotes quantities of nominal bonds, as b_t denoted real bonds previously. Q_t is the price of a nominal bond in cash units (similar to q_t). The implied return on nominal bonds

$$1 + i_t = \frac{1}{Q_t}$$

is the *nominal interest rate*. The growth in the price level is the *inflation rate*:

$$1 + \pi_t = \frac{P_t}{P_{t-1}}.$$

The *real interest rate* is the return on an investment in the nominal bond in terms of real goods. As you can see below, it coincides with the ratio of nominal interest and next-period realized inflation:

$$1 + r_t = \frac{1/P_t}{Q_{t-1}/P_{t-1}} = \frac{1 + i_{t-1}}{1 + \pi_t}.$$

The balance of money held by households in the beginning of period t is M_t, and $m_t = M_t/P_t$ denote its real value (or the amount of consumption goods it can purchase).

We return to our two-period setup, but it is easier for the exposition to discuss each period separately.

6.1.1. Period One

At the beginning of period one, the government redeems B_0 maturing bonds for currency, which moves to the hands of households. Then, the government announces lump-sum taxes τ_1 and public spending g_1, both stated in units of consumption goods. The real primary surplus is $s_1 = \tau_1 - g_1$. Taxes and public spending are paid with cash. Therefore, by running a primary surplus, the government removes money from circulation. The borrowing constraint faced by the government is

$$B_0 = P_1 s_1 + (M_1 - M_0) = P_1 s_1 + \Delta M_1. \tag{6.1}$$

(The world ends in $t = 2$, so households do not buy new bonds.) The interpretation of (6.1): the cash the government uses to redeem bonds B_0 (left side) is either removed from circulation by surpluses $P_1 s_1$ (right side) or added to households' stock of money ΔM_1 (right side).

The term ΔM_t is called *seignorage*. It is the revenue obtained by the government for having the right to issue money. We can split the real revenue raised through seignorage between the growth rate of real money stock and a term representing the *inflation tax*:

$$\text{Seignorage} = \frac{\Delta M_t}{P_t} = \underbrace{\Delta m_t}_{\substack{\text{Real Money}\\ \text{Growth}}} + \underbrace{\frac{\pi_t}{1+\pi_t} m_{t-1}}_{\substack{\text{Inflation}\\ \text{Tax}}}.$$

The inflation tax represents the loss in purchasing power of money over time.

Adding $+M_0$ on both sides of (6.1) yields

$$V_0 \equiv M_0 + B_0 = P_1 s_1 + M_1. \tag{6.1a}$$

V_0 is the amount of circulating cash after the government redeems bonds at the beginning of period one, comprising currency households brought from period zero M_0, and new cash introduced from bond redemptions B_0. You can regard V_0 as the "total" size of public debt. The government must "repay" this amount either by running surpluses, or by relying on households to hold currency at the end of period one. Why households would accept holding cash $M_1 > 0$ when they do not demand bonds $B_1 = 0$ is the topic of the next section.

6.1.2. Active Monetary vs Active Fiscal Models

The introduction of nominal debt blurs the connection between budget constraints and default-averting surpluses. In the models with real debt considered in the previous chapters, public budget constraints assigned a required level of primary surplus to avoid a public default:

$$b_0 = s_1. \tag{6.2}$$

Since the government cannot create consumption goods, it must tax households to procure them, and then repay bondholders. Nominal debt, on the other hand, redeems for currency, a commodity the government *can* create and thus never needs to default on - regardless of s_1. Therefore, in principle, there is no minimal primary surplus the government needs to announce to prevent a default. At the beginning of the period one, the government redeems $10 dollars in debt (or $100, or $1,000) by issuing currency, and then announces taxes of one good per household (or two, or three). Cash obligations do not restrict surpluses.

The budget constraint *does* constraint surpluses, *given the price level.* The "given the price level" clause is a major difference between the purely real economy models of the previous section, and the monetary models we study in this one. To understand that difference, divide both sides of (6.1) by the price level:

$$\frac{B_0}{P_1} = s_1 + \frac{\Delta M_0}{P_1}. \tag{6.1b}$$

Real debt = real public income. Comparing the budget constraint in the monetary model (6.1b) with its real-model counterpart (6.2), we see two differences. First is the seignorage term on the right, which we discuss later. Second, and most importantly, the price level P_1 now shows up on the denominator on the left-hand side, which makes real debt no longer a predetermined variable. A higher price level in period one reduces the real value of nominal bonds. The government owes *less* to bondholders, in terms of consumption goods. In that case, the budget constraint (6.1b) says that the associated primary surplus (plus seignorage) is smaller.

Whether (6.1b) pins public revenue given prices, or prices given public revenue is a question of large debate in the literature, and divides the set of monetary-fiscal models in two. *Active monetary, passive fiscal models* have been the most common assumption. In it, the government observes the price level P_1 (whose equilibrium value is determined elsewhere in the model), and announces enough surpluses to guarantee that the budget constraint (6.1b) holds. The reason behind the name "active monetary" will become clear when we discuss money demand and the price level in equilibrium. *Active fiscal, passive monetary* models are the basis for the fiscal theory of the price level, a (mostly) more recent approach. In it, the government announces primary surpluses regardless of the price level - we have seen that it *can* do this. Households' willingness to hold on to cash, and their own budget constraint then imply that the equilibrium price level satisfies the public budget constraint (6.1b).

In this section, we focus on active monetary models. We study the fiscal theory of the price level in the next chapter.

6.1.3. Period Zero

We move one period backwards. In period zero, the story is similar. The key difference from period one is that, in period zero, the government can sell nominal debt - which also removes money from circulation. Households start with M_{-1} units of currency and B_{-1} nominal bonds. They pay $Q_0 B_0$ in cash to the government in exchange for B_0 public bonds. The budget constraint in period zero becomes

$$B_{-1} = Q_0 B_0 + P_0 s_0 + \Delta M_0. \tag{6.3}$$

On the left side, currency put in circulation through bond redemption; on the right, where it flows to: new bond purchases, tax payments (net of public spending), or households' pockets. Like before, we can re-write this budget constraint in terms of the total size of government debt after bond redemption, $V_t = B_t + M_t$:

$$V_{-1} = Q_0 V_0 + P_0 s_0 + (1 - Q_0) M_0. \tag{6.3a}$$

The government can "repay" debt by issuing more debt at a price Q_0 or by running a primary surplus. The last term

$$(1 - Q_0) M_0 = \frac{i_0}{1 + i_0} M_0$$

represents the convenience yield obtained by the government for "selling" money, a debt-like asset that pays no interest. When $i_0 = 0$, bonds and currency become economically identical, and the convenience yield vanishes.

6.2. Money Demand and the Equation of Exchanges

6.2.1. Households and Preference for Liquidity

In frictionless models, households do not demand currency, because currency does not pay interest. Nonzero demand for currency therefore requires the existence of *frictions* in the economy that renders money valuable. These frictions are usually motivated by the idea that, in reality, money has some special "quality" that facilitates trade. If you inadvertently come cross that handcrafted bow tie you were looking for, you cannot instantly sell your bonds to pay the tailor (or just transfer them to him/her); you have to have money on your wallet. Admittedly, as payment technologies evolve, justifying such frictions becomes harder. Nevertheless, because $MV = PY$ continues to be heavily employed in the academic literature, we proceed under the assumption that these frictions are well justified.

Capture households' preference for liquidity through the money-in-the-utility function formulation. Endowment economy, no production. Natural borrowing limit.

$$\begin{aligned} \underset{c,M,B_0}{\text{Max}} \quad & u(c_0) + h(m_0) + \beta\left[u(c_1) + h(m_1)\right] \\ Q_0 B_0 + M_0 + P_0 c_0 &\leq B_{-1} + M_{-1} + P_0(y_0 - \tau_0) \\ P_1 c_1 + M_1 &\leq B_0 + M_0 + P_1(y_1 - \tau_1) \\ c, M &\geq 0. \end{aligned} \tag{6.4}$$

Function h satisfies usual assumptions: twice differentiable, increasing and concave. Recall that $m_t = M_t/P_t$: households have preferences for *real* holdings of money.

First-order condition for public bonds:

$$Q_0 u'(c_0) = \beta \frac{u'(c_1)}{1+\pi_1} \qquad \Longrightarrow \qquad u'(c_0) = \beta(1+r_1)u'(c_1) \tag{6.5}$$

Equation (6.5) is the same first-order condition we find in the real-bond model.

First-order condition for money balances in period zero:

$$u'(c_0) = h'(m_0) + \beta \frac{u'(c_1)}{1+\pi_1} \tag{6.6}$$

Interpration: marginal utility cost of increasing money balance (left side) equals marginal utility benefit (right side).

First-order condition for money balances in period zero:

$$h'(m_1) = u'(c_1) \tag{6.7}$$

From (6.5) and (6.6):

$$h'(m_0) = (1 - Q_0)u'(c_0) \tag{6.8}$$

The marginal utility of one additional real unit of currency equals the utility cost of the convenience yield. Like we did for public budget constraints, we can re-write households' constraints and its entire optimization problem in terms of total assets $V_t = B_t + M_t$ and the conveniece yield, as follows.

$$\begin{aligned} &\operatorname*{Max}_{c,M,B_0} \quad u(c_0) + h(m_0) + \beta\left[u(c_1) + h(m_1)\right] \\ &Q_0V_0 + (1 - Q_0)M_0 + P_0c_0 \leq V_{-1} + P_0y_0 \\ &P_1c_1 + M_1 \leq V_0 + P_1y_1 \\ &c, M \geq 0. \end{aligned} \tag{6.9}$$

6.2.2. Central Bank and Equilibrium

The monetary authority (or the *Central Bank*) inelastically supplies money in both periods, M_0 and M_1. We also fix public spending g, and assume passive fiscal policy: the government chooses taxes τ to satisfy the budget constraint at the equilibrium price level.

In equilibrium, $y_t = g_t + c_t$. Therefore:

$$1 + r_1 = \frac{1 + i_0}{1 + \pi_1} = \frac{u'(y_0 - g_0)}{\beta u'(y_1 - g_1)} \tag{6.10}$$

$$h'(m_0) = \frac{i_0}{1 + i_0}u'(y_0 - g_0) \tag{6.11}$$

$$h'(m_1) = u'(y_1 - g_1) \tag{6.12}$$

(in (6.11) we have replaced $1 - Q_0 = i_0/(1 + i_0)$).

Expression (6.10) determines the real interest rate. Like in the real economies of the previous sections, the interest rate is marginal rate of substitution between consumption in periods zero and one. Importantly, Central Bank activity does not affect the real interest rate - a property of models in which prices are flexible.

Expressions (6.11) and (6.12) are the first versions we encounter of the celebrated *equation of exchanges*:

$$M_tV_t = P_ty_t. \tag{6.13}$$

As originally written down by Irving Fisher, the equation of exchanges (or simply MV=PY) posits that demand for cash (M) balances is directly proportional to the nominal volume of transaction, which we usually approximate using aggregate nominal income (Py). The scaling constant V is the *velocity of money.* The name follows from the (somewhat loose) interpretion of the PY-to-M ratio as the number of times agents use the same unit of currency to make a purchase.

Equation (6.12) pins down P_1. Equation (6.11) jointly determines P_0 and the i_0. The fact that the Central Bank choice of money supply determines the price level justifies the designation of the model as an "active monetary" model.

While our framework has microfounded some version of $MV = PY$, equation (6.11) does not necessarily lead to a version of (6.13). The exercises consider a particular case of isoelastic u and h in which it does. Yet, in general case, (6.11) implies a positive relationship between real money balances $m_0 = M_0/P_0$ and real income y_0 (even if the latter is shifted by public spending g_0). Fixing public spending, higher income y_0 corresponds to higher consumption in equilibrium, therefore wealthier households. Wealthier households demand more real money balances. Now you can see how the money-in-the-utility-function formulation captures the idea that more income asks for larger holdings of cash.

In the case of period zero, expression (6.11) also shows that, for a fixed y_0, demand for real holdings of currency are decreasing (thus "velocity" is increasing) in the nominal interest rate. Intuitively, higher interest rates increases the opportunity cost of holding on to currency.

Monetarism and backing theories

6.2.3. Seignorage

Central bank activity and seignorage as a means of financing government's deficits. Real seignorage depends households' acceptance of cash holdings (velocity). Start with a general equation of exchanges, in which velocity depends on nominal interest:

$$M_t V(i_t) = P_t y_t$$

Taking difference (assume fixed interest rate $i_t = i$):

$$\Delta(M_t V(i)) = \Delta(P_t Y_t) \implies V(i)\Delta M_t = P_t \Delta y_t + y_t \Delta P_t$$

Let $g_t = \Delta y_t / y_{t-1}$ be real income growth. Manipulating the expression above yields

$$V(i)\frac{\Delta M_t}{P_t y_t} = \frac{g_t}{1+g_t} + \frac{\pi_t}{(1+g_t)(1+\pi_t)}.$$

We can roughly simplify the denominators on the right to one, which leads to the convenient expression

$$\frac{\Delta M_t}{P_t y_t} = \frac{g_t + \pi_t}{V(r+\pi_t)}. \tag{6.14}$$

The left side of the (6.14) is real seignorage, as a share of aggregate real income.

How does the seignorage revenue depend on the inflation choosen by the Central Bank? We take logs (we are interested in relative, not absolute changes), and differentiate (6.14) with respect to π_t to find

$$\frac{\partial \log M_t/(P_t y_t)}{\partial \pi_t} = \frac{1}{g_t + \pi_t} - \frac{\partial \log V(r+\pi_t)}{\partial \pi_t} \tag{6.15}$$

If the elasticity of money velocity is large, more inflation can reduce the seignorage term, as households run from cash - and that effect exceeds the inflation tax.

To find the *revenue-maximizing* inflation rate, equate (6.15) to zero. We find

$$g_t + \pi_t = \left[\frac{\partial \log V(r+\pi_t)}{\partial \pi_t}\right]^{-1}.$$

The revenue-maximizing inflation rate depends negatively on the elasticity of velocity.

6.3. Cagan's Model of Hyperinflations

We begin to study monetary-fiscal interactions.

Cagan (1956) considers hyperinflation events, which he defines as monthly inflation rates superior to 50%. Cagan argues that, during hyperinflation episodes, the equilibrium values of real variables are independent of variation in the price-level. In the context of a monetary model, one can then abstract from variation in the real interest rate r_t and in aggregate income y_t.

Cagan posits a (log) money demand equation of the format

$$\hat{m}_t + \eta i_t = p_t, \tag{6.16}$$

where $\hat{m}_t = \log M_t$ (not to be confused with $m_t = M_t/P_t$), and $p_t = \log P_t$. The term ηi_t captures money velocity, which is a function of nominal interest, and hence of real interest (constant, we can normalize to zero) and expected next-period inflation π^e_{t+1}. Parameter η is the elasticity of (log) money velocity with respect to inflation $\partial \log V / \partial \pi$.

Cagan assumes *adaptive expectations*, meaning that expected inflation depends on past inflation rates. For simplicity, we assume that it coincides wiht current inflation:

$$\pi^e_{t+1} = p_t - p_{t-1}.$$

(Since p_t is log price level, $p_t - p_{t-1} \approx \pi_t$.)Equations (6.16) becomes

$$\hat{m}_t + \eta(p_t - p_{t-1}) = p_t. \tag{6.17}$$

or, yet:

$$p_t = \frac{\eta}{\eta - 1} p_{t-1} - \frac{1}{\eta - 1} m_t.$$

If $\eta > 1$ (velocity highly elastic), inflation can be driven by momentum. Inflation leads to a decline in velocity, which induces more inflation. Additionally, as velocity declines, seignorage generates less and less revenue to the government.

6.4. Unpleasant Monetarist Arithmetic

Throught their seminal paper, Sargent and Wallace (1981) were the maybe first to consider the implications of active fiscal policy to price level determination. Active fiscal policy means that, instead of adjusting surpluses s to satisfy (6.1b), the government fixes s. To prevent a government default, the Central Bank at some point must increase money supply enough to generate large enough seignorage revenues. The "unpleasant" tautology follows from the fact that, the longer the Central Bank waits to monetize public debt, the more inflation is required to prevent the default.

Like the original paper, it is easier to cast the model using real debt (although the concepts holds with nominal debt too - see the exercises). Real debt is paid with currency at the beginning of each period, so the government's budget constraints are

$$\begin{aligned} P_0 b_0 &= P_0 q_0 b_1 + P_0 s_0 + \Delta M_0 \\ P_1 b_1 &= P_1 s_1 + \Delta M_1. \end{aligned} \tag{6.18}$$

The present-value budget contraint is

$$b_0 = \left(s_0 + \frac{\Delta M_0}{P_0} \right) + q_0 \left(s_1 + \frac{\Delta M_1}{P_1} \right)$$

Replacing the expression for seignorage (6.14) yields:

$$b_0 = \left(s_0 + \Delta m_0 + \frac{\pi_0}{1+\pi_0} m_{-1} \right) + q_0 \left(s_1 + \Delta m_1 + \frac{\pi_1}{1+\pi_1} m_0 \right). \qquad (6.19)$$

Sargent and Wallace assume the equation of exchanges (6.13) holds in both periods, with the same constant velocity $V_0 = V_1$ and aggregate output $y_0 = y_1$. Consequently, demand for real holdings of money is constant: $\Delta m_0 = \Delta m_1 = 0$, and seignorage coincides with the inflation tax.

We start with an equilibrium (M, P), and consider a different equilibrium (M', P'). In this second equilibrium, the Central Bank decides to reduce money supply in period zero: $M_0' < M_0$. Active fiscal policy remains unchanged: $s' = s$.

1. By MV=PY, $P_0' < P_0$, so $\pi_0' < \pi_0$.

2. Lower inflation means lower inflation tax in period zero. Given constant surpluses, public debt at the end of period zero is larger in the second equilibrium.

3. With a larger debt, in period one the Central Bank must raise money supply to generate enough seignorage revenues and prevent a default: $M_1' > M_1$.

4. By MV=PY, $P_1' > P_1$.

Lower money supply in period zero reduces inflation in period zero, but increases it in period one. Additionally, (6.19) and $q_0 < 1$ imply that the increase in inflation rate in period one required to prevent a default is larger than its period-zero decline. The longer the Central Bank takes to monetize debt, the larger the required issuance of money - and thus the larger the ensuing rise in inflation.

Exercises

Exercise 6.1. Suppose the Central Bank announces at the beginning of period zero that it will double money supply, compared to what agents previously expected: $M' = 2 \times M$. Based on the micro-founded money demand equations (6.11) and (6.12), how will the new policy affect: (a) the price level in each period; (b) the inflation rate in each period (you can take as given the price level in $t = -1$, P_{-1}); (c) the interest rate in period zero i_0?

Exercise 6.2. Suppose $g_0 = 0$. Assume u and h are both isoelastic period utility functions, *i.e.*:

$$u(x) = h(x) = \frac{x^{1-\frac{1}{\gamma}} - 1}{1 - \frac{1}{\gamma}}.$$

(a) Using equilibrium condition (6.11), and define a velocity of money variable V_0 that satisfies the traditional equation of exchanges

$$M_0 V_0 = P_0 y_0.$$

When we increase the elasticity of intertemporal substituion γ, does the velocity get more or less responsive to variation in the nominal interest rate?

(b) Repeat (a) for the equilibrium condition (6.12). What is the velocity of money in period one?

Exercise 6.3. Suppose $u(c) = \log(c)$ and $h(m) = \log(m)$. Use your result from exercise Exercise 6.2. to compute money velocity. Let $1+g_M = M_1/M_0$ be money supply growth. Use equilibrium conditions (6.11)-(6.12) to compute nominal interest rate i_0 as a function of real interest r_1 and g_M.

Exercise 6.4. Cagan's model with forward-looking expected inflation (Kenneth and Rogoff).

Exercise 6.5. Consider the unpleasant arithmetic environment, and the experiment of a money supply reduction in period zero by the Central Bank, but suppose that debt is nominal instead of real.

(a) How does the present-value budget constraint (6.19) change if we consider nominal instead of real debt?

(b) Suppose that initially (*i.e.*, before the announcement of the change in money supply) the equilibrium values of fiscal surpluses, money stock and price level are the same, in $t = 0$ and $t = 1$. The reduction in money supply calls for a greater or a smaller increase in period-one price level? Explain intuitively.

Chapter 7

The Fiscal Theory of the Price Level

7.1. The Basic FTPL Model

This chapter is mostly based on Cochrane (2023).

As in the previous chapter, it is easier to first discuss the fiscal theory of the price level in a one-period setting, and then generalize the intuition to multiple periods. So we start by focusing in period one.

7.1.1. Period One: Fiscal Theory in a One-Period Setup

We consider the same environment of chapter 6. Households bring B_0 nominal public bonds from period zero. At the beginning of period one, the government redeems these bonds for currency, which moves to the hands of households, and announces a *lump-sum* tax of $\tau_1 > 0$. For simplicity, there is no public spending. The market for consumption goods opens, and the price level P_1 forms. Finally, households pay taxes using cash.

Fiscal policy is *active*. The government chooses a *fixed* primary surplus that does not respond to other economic variables. Because it does not demand goods, the primary surplus is $s_1 = \tau_1 > 0$. There is nothing wrong or unnatural in having the government choose real taxation independently of public nomional debt. Because nominal bonds redeem for currency - a commodity the government can create at zero cost -, debt obligations do not constraint real surpluses. The government's budget constraint

$$B_0 = P_1 s_1 + \Delta M_1, \tag{7.1}$$

has a different interpretation, which is: the cash used to redeem bonds B_0 must either be retired by public taxation $P_1 s_1$ or voluntarily absorbed by households $\Delta M_1 = M_1 - M_0$ (M_t denotes stock of currency).

Moving to the household side, we adopt an environment similar to that the previous section, except that households have no preference for liquity. We want what one can call a "neoclassical" model, in which money has no special property. As such, contrary to chapter 6, households' utility

$$u(c_0) + \beta u(c_1)$$

does not depend on real money holdings $m_t = M_t/P_t$. Other than that, each household has a fixed endowment of y_1 consumption goods, and chooses its consumption level c_0 and money holding M_1 subject to the budget constraint

$$P_1 c_1 + \Delta M_1 = B_0 + P_1 y_1 - P_1 s_1. \tag{7.2}$$

The right-hand side of (7.2) implies that the household cannot spend the entirety of its available cash $B_0 + P_1 y_1$. At the end of period one, it must have $P_1 s_1$ units of currency left to pay out the government.

We are interested in determining the equilibrium price level P_1. In equilibrium, the market for goods clears: $c_1 = y_1$. In addition, because money offers no value to households, their demand for cash is zero: $M_1 = 0$. For a similar reason, households do not bring money from period zero $M_0 = 0$. Hence, $\Delta M_1 = 0$. Replacing these conditions into (7.2) leads to an expression that pins down the equilibrium price level:

$$\frac{B_0}{P_1} = s_1. \tag{7.3}$$

The emergence of the equilibrium price level as the solution to (7.3) follows directly from household optimal behavior, and equilibrium in the goods market. When acting on the market for goods, households choose to set aside *exactly* the volume of cash necessary to pay taxes.

$$\underbrace{B_0 + P_1(y_1 - c_1)}_{\substack{\text{Currency left after} \\ \text{market for goods closes}}} = P_1 s_1.$$

Setting aside less than $P_1 s_1$ units of currency forbids a household from paying all due taxes. Setting aside more than $P_1 s_1$ forces the households to hold currency, which offers no value. And what is the volume of currency households set aside in the market for goods? Equilibrium in the goods market implies that any volume of currency accumulated by a household for selling a good must be spent by another household who buys it: $P_1 y_1 = P_1 c_1$. Hence, the volume of currency on the pockets of households after the market for goods closes equals B_0 - the volume of currency introduced by the government through bond redemption. Equation (7.3) emerges.

We can interpret the equilibrium in terms of movements in "aggregate demand". Let P_1 be the solution to (7.3). If the price level was $P_1' > P_1$, so that $B_0 < P_1' s_1$, households would not have enough currency to pay taxes, which would force them to reduce demand for goods. As all households attempted to reduce consumption, the price level would decline until it reached P_1. The analogous story explains why $P_1' < P_1$ cannot be the equilibrium price level.

We can also interpret the equilibrium in terms of a nominal volume relative to a real volume. The fiscal theory of the price level is a *backing theory*, similar to active monetary models based on MV=PY. The nominal volume is the stock of nominal debt B_0. The real volume is the flow of (real) primary surpluses. For the same announcement of primary surpluses, larger stocks of nominal debt lead to greater price levels.

7.1.2. Period Zero: Fiscal Theory in a Two-Period Setup

We move one period backwards to understand price level formation in an interperiod model. The price of a nominal bond in period zero is Q_0. The budget constraints for the government and households become

$$Q_0 B_0 + P_0 s_0 + \Delta M_0 = B_{-1} \tag{7.4}$$

$$Q_0 B_0 + P_0 c_0 + \Delta M_0 = B_{-1} + P_0 y_0 - P_0 s_0. \tag{7.5}$$

In equilibrium, $y_0 = c_0$ and $M_{-1} = M_0 = 0$. Constraints (7.4) and (7.5) lead to the following flow equation for real public debt:

$$\frac{B_{-1}}{P_0} = s_0 + \frac{Q_0 B_0}{P_0} \tag{7.6}$$

Like in period one, surpluses soak up currency, but now so do the sales of new public bonds, which raises $Q_0 B_0$ units of currency in revenues to the government.

The usual household consumption-savings problem leads to the Euler equation

$$u'(c_0) Q_0 = \beta \frac{P_0}{P_1} u'(c_1).$$

Henceforth, we assume constant output $y_0 = y_1$, which implies $c_0 = c_1$ and that the real interest rate is $1/\beta$ in equilibrium. (If you find $y_0 = y_1$ too hard of an assumption, call $1/\beta$ the real interest rate and proceed with no restrictions on y.) In any case, the equilibrium price of the nominal bond is:

$$Q_0 = \frac{1}{1+i_0} = \beta \frac{P_0}{P_1}. \tag{7.7}$$

Replacing (7.7) in the flow equation (7.6) yields

$$\frac{B_{-1}}{P_0} = s_0 + \beta s_1. \tag{7.8}$$

Equation (7.8) generalizes (7.3). It is known as the *valuation equation of public debt*. Given the sequence for primary surplus s and the predetermined size of public debt B_{-1}, it determines the price level in period zero P_0.

Equilibrium in period zero translates the same intuition as that of period one. Households run towards or away from currency until they have just enough to pay for their taxes. That is the interpretation of equation (7.6) - note that (7.6) is an *equilibrium conditions*, not a budget constraint. Such household behavior, coupled with a predetermined volume of public debt, determines the price level. The key difference we find in period zero is that households also set aside cash to pay for new nominal bonds. How much? The equilibrium we computed in period zero shows that the real revenue raised by the government with new bond sales equals βs_1. Hence the connection between period-one surpluses and period-zero price level.

7.1.3. Fiscal and Monetary Policy

Fiscal policy: changes to $s = (s_0, s_1)$ lead to changes in the price level in both periods.

Connection between debt/deficits and inflation? Not necessarily: households can expect future surpluses that repay large debt at a given price level. Be careful with the branding "active fiscal"!

Monetary policy: changes to B_0 unaccompanied by changes to primary surpluses. Since

$$\frac{Q_0 B_0}{P_0} = \beta s_1,$$

and since P_0 is determined by the valuation equation (7.8), if the Central Bank sells additional units of B_0, then Q_0 must fall to the point that bond sales revenue $Q_0 P_0$ remains unchanged. Thus, the Central Bank faces a *unit-elastic* demand for public bonds. Instead of fixing bond sales B_0, it can fix Q_0 (and nominal interest rate) and elastically offer B_0 - a horizontal supply curve of public bonds.

Effect of bond sales on inflation? The Fisherian effect: higher nominal interest leads to higher inflation. Intuition?

7.2. Expected and Unexpected Inflation

Environment with *surplus risk*. Primary surplus (in both periods) is a random variable which households do not know in the previous period.

In the presence of uncertainty, the price of a nominal bond becomes

$$Q_0 = \frac{1}{1+i_0} = \beta E_0 \left[\frac{P_0}{P_1}\right]. \tag{7.9}$$

Replacing (7.9) on (7.6) gives a new version of the valuation equation:

$$\frac{B_{-1}}{P_0} = s_0 + \beta E[s_1]. \tag{7.10}$$

Equation (7.9) shows that monetary policy - the setting of nominal interest by the Central Bank - pins down *expected inflation*.

Let

$$\Delta E_t = (E_t - E_{t-1})$$

be the innovation operator. For a random variable whose value is unknown prior to period $T \geq t$,

$$\Delta E_t x_T = E_t x_T - E_{t-1} x_T$$

captures the revision in expectation of x_T. In particular, when $T = t$:

$$\Delta E_t x_t = x_t - E_{t-1} x_t$$

captures the unexpected component of the realization of x_t relative to expectation in $t-1$.

Taking the innovation operator in (7.3):

$$\frac{B_0}{P_0}\Delta E_1 \left(\frac{P_0}{P_1}\right) = \frac{B_0}{P_0}\Delta E_1 \left(\frac{1}{1+\pi_1}\right) = \Delta E_1 s_1$$

Unexpected inflation in period one is pinned down by fiscal policy. The same is true in period zero:

$$\frac{B_{-1}}{P_{-1}}\Delta E_0 \left(\frac{1}{1+\pi_0}\right) = \Delta E_0 s_0 + \beta \Delta E_0 s_1.$$

7.3. A Fiscal Theory of Monetary Policy

We explore two extensions that make monetary-policy interactions more interesting.

7.3.1. A Price-Level Target

Instead of fixing a level of primary surplus s_1, the government observes the level of debt in period zero and establishes a *price level target* P_1^*. It then sets primary surplus to guarantee that price level in equilibrium:

$$s_1 = \frac{B_0}{P_1^*} \tag{7.11}$$

In period zero:

$$\frac{B_{-1}}{P_0} = s_0 + \beta \frac{B_0}{P_1^*}. \tag{7.12}$$

Given the price target, the government can finance a period-zero deficit (s_0 down) by issuing new bonds (B_0 up) without affecting the price level. The increase in bond issuance comes accompanied by the implicit promise of an increase in surpluses in period one, so that (7.11) holds.

Monetary-fiscal interaction: suppose the Central Bank raises B_0 to reduce bond price Q_0. In period one, the government raises s_1 accordingly: price level P_1^* unchanged. Thus, with the price-level target, monetary policy interacts with fiscal policy! In period zero, P_0 *declines* by (7.12). Inflation from period zero to one increases, implying that nominal interest increases as well. But the fact that the price level declines in period zero gives a perhaps comforting result: an interest rate rise reduces *current* inflation P_0/P_{-1}.

7.3.2. Model with Long-Term Debt

Second pathway to generate higher interest leading to lower inflation. Instead of one-period bonds only, bonds can have any maturity. Q_t^n is the price of a bond that promises the delivery of one unit of currency after n periods. (So far, we have dealing only with $Q_t^{n=1}$.)

The government cannot sell bonds maturing after period one. Therefore equilibrium condition (7.3) continues to hold. In period zero, the flow equation of government debt is

$$B_{-1}^0 = P_0 s_0 + Q_0^1 (B_0^1 - B_{-1}^2)$$

The parenthesis term on the right represents the amount of currency the government retires by selling *additional* nominal bonds maturing in period one. We can re-write it in real terms

$$\frac{B_{-1}^0 + Q_0^1 B_{-1}^2}{P_0} = s_0 + \frac{Q_0^1 B_0^1}{P_0} \tag{7.13}$$

Replacing (7.3) gives the new version of the valuation equation in period zero:

$$\frac{B_{-1}^{0}+Q_{0}^{1}B_{-1}^{2}}{P_{0}}=s_{0}+\beta s_{1} \tag{7.14}$$

The left-hand side contains the real *market-value* of public debt in the beginning of period zero.

Suppose that the Central Bank fixes a target for Q_0^1 (interest rate), and sells one-period bonds accordingly. Primary surpluses s remain constant. What are the effects of an increase in nominal interest, or a decline in Q_0^1? Higher interest reduces the price of nominal bonds, and hence the market value of public debt - the left-hand side of (7.14). This is the major difference compared to the one-period bond case. The equilibrium price level in period zero P_0 declines. We again see tighter monetary policy lead to a decline in current inflation.

Nevertheless, to raise nominal interest, the Central Bank increases sells more one-period bonds (an exercise asks you to prove that claim), leading to more inflation in period one. Higher inflation validates the Fisher relationship.

7.4. Observational Equivalence

Is it possible to test active vs passive fiscal policy? To answer that question, we compare the equilibrium conditions that arise from the active monetary models of chapter 6 with the equilibrium conditions from the FTPL. To do a valid comparison, we need to relax the assumption the households do not value currency. Therefore, consider that households have preferences and solve a consumption-savings problem similar to those assumed in the previous chapter:

$$\begin{aligned} &\underset{c,M,B_0}{\text{Max}} \quad u(c_0)+h(m_0)+\beta\left[u(c_1)+h(m_1)\right] \\ \text{s.t.} \quad & Q_0B_0+M_0+P_0c_0\le B_{-1}+M_{-1}+P_0(y_0-\tau_0) \\ & M_1+P_1c_1\le B_0+M_0+P_1(y_1-\tau_1) \\ & c,M\ge 0. \end{aligned} \tag{7.15}$$

Function u and h satisfy the usual assumptions (increasing, concave, twice differentiable). We call $m_t=M_t/P_t$ real money holdings. The government charges lump-sum taxes $\tau=(\tau_0,\tau_1)$ and demands $g=(g_0,g_1)$ consumption goods in the market. The difference $s=\tau-g$ is the primary surplus series.

Computing the first-order conditions of problem (7.15) and imposing that all markets clear yields the equilibrium conditions:

$$u'(y_0 - g_0) = \beta(1 + r_1)u'(y_1 - g_1) \tag{7.16}$$

$$h'(m_0) = \frac{i_0}{1 + i_0}u'(y_0 - g_0) \tag{7.17}$$

$$h'(m_1) = u'(y_1 - g_1) \tag{7.18}$$

Equations (7.16)-(7.18) are the same equilibrium conditions as (6.10)-(6.12). Since budget constraints are also the identical in both models, we conclude that the equilibrium generated by active monetary and active fiscal models are the same. This result is referred to as *observational equivalence*.

Active fiscal: $B/P = PV(s)$ determines the price level. The Central Bank provides an "elastic" currency, to satisfy trading needs. Active fiscal: The Central Bank fix the supply of money, and $MV = PY$ determines the price level. The government choses surpluses to satisfy $B/P = PV(s)$ at the price level effectively set by the Central Bank.

Exercises

Exercise 7.1. Public primary surplus in period one follows the distribution below.

$$s_1 = \begin{cases} \bar{y}_1 + z & \text{with prob. } \frac{1}{2} \\ \bar{y}_1 - z & \text{with prob. } \frac{1}{2} \end{cases}$$

In period zero, the government inherits a debt of B_{-1} nominal bonds, and charges a deterministic surplus of s_0 consumption goods. Compute the price level in period zero P_0, expected period-one inflation $E_0 P_0/P_1$ and the unexpected component of inflation in period one $\Delta E_0(P_0/P_1)$.

Exercise 7.2. Consider again the environment with long-term debt. We want to show that an increase in nominal interest lowers current inflation, but raises future inflation.

(a) Assume $B^0_{-1} > 0$. Use equation (7.14) to argue that, as the Central Bank raises nominal interest, Q^1_0/P_0 falls.

(b) Compute the real revenue raised by the government for selling new debt. Why is real revenue unchanged? To sustain a higher nominal interest, does the Central Bank sell more or less one-period debt in period zero?

(c) Use the valuation equation to argue that, after falling in period zero, inflation increases in period one.

Exercise 7.3. (Stock as Money.) The equilibrium condition (7.3) that yields the price level in a FTPL model is an instance of an asset pricing equation: asset price = discounted dividends. Primary surpluses act like "dividends" to public bonds, as bondholders can use them to pay for taxes. The higher the taxes (or primary surplus), the more valuable the nominal bond becomes. To formalize this analogy, in this exercise we consider an alternative monetary model, in which agents use stocks of a representative firm as currency.

There is no government. Each household has access to y_t consumption goods in the form of home production. The representative firm owns a technology that freely yields d_t consumption goods per period - this is known as a *Lucas tree*, following Lucas (1978). The firm sells these goods to households, in exchange for its own stock. Households also use stocks of the representative firm to carry out trade among themselves. The price of one consumption good in terms of stocks is P_t.

(a) Write the market-clearing condition of the consumption goods market in each period.

(b) Suppose households enter period one with X_0 stock units. Write the household's budget constraint in period one, and then in period one. Importantly, there are no stock splits or repurchases between periods zero and one. If the household ends period zero with X_0 stocks, that is the amount of stock it will begin period one with.

(c) Using the utility function $u(c_0) + \beta u(c_1)$, where u is concance, differentiable and increasing, find the Euler equation for firm stocks. Then, assuming that the aggregate endowment series $y + d$ is constant, show that

$$\frac{1}{P_0} = \beta \frac{1}{P_1}. \tag{7.19}$$

Intepret the equilibrium condition above.

(d) Impose the market-clearing conditions and (7.19) on households' budget constraints to find the new valuation equations determining the price level:

$$\frac{X_0}{P_1} = d_1 \qquad \text{and} \qquad \frac{X_{-1}}{P_0} = d_0 + \beta d_1$$

The stock price in terms of goods - the inverse of the price level - coincides with discounted dividends (interpreting as dividends the output sold by the firm, or its profit). Interpret why a price level $P_1^* > P_1$ (where P_1 is determined by the equation above) cannot be an equilibrium price.

Chapter 8

Fiscal Multipliers

Money multiplier = effect of a change in public spending or investment on national income/output. We focus on public spending.

$$\frac{\Delta y}{\Delta g}$$

There is no single fiscal multiplier. In practice, multiplier depends on many factors, such as the time schedule of the purchases and whether it is anticipated by households.

8.1. Keynesian Cross

The Keynesian Cross model is not a microfounded model, but it seems to base a lot of policy discussions. The model posits a consumption function

$$c = \alpha + \phi x, \tag{8.1}$$

where c represents aggregate consumption and x represents aggregate income. Since the model is static, we supress time subscripts. Parameter ϕ represents the *marginal propensity to consume* and plays a critical role in the model.

In equilibrium, aggregate output y equals the sum of household and public spending:

$$y = c + g = \alpha + g + \phi x. \tag{8.2}$$

Assuming a closed economy, national income coincides with national output: $y = x$. We

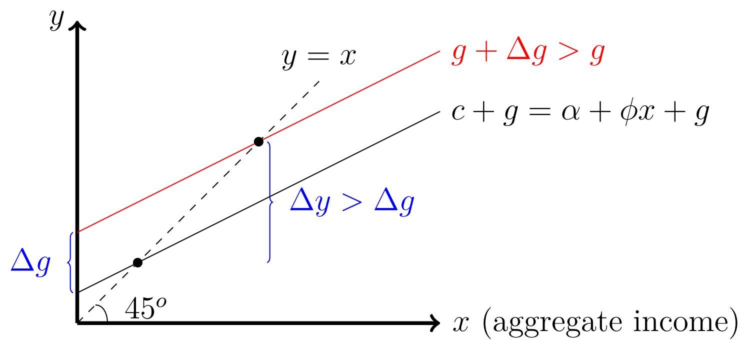


Figure 8.1: Equilibrium in the Keynesian Cross Model

then have the following solution to the Keynesian Cross model:

$$y = \frac{\alpha + g}{1 - \phi}. \tag{8.3}$$

The black curves in figure 8.1 represents the equilibrium described by the model, and explains why we call it Keynesian "Cross". We find the equilibrium in the point in which the aggregate demand curve $c + g$, expressed as a function of national income, crosses the identity line output income = output output.

The fiscal multiplier is

$$\frac{\partial y}{\partial g} = \frac{1}{1 - \phi}.$$

Higher public spending Δg boosts aggregate demand $c + g$, which leads to an increase in output, and therefore an increase in income x. In turn, higher income further boosts aggregate demand $c + g$ through the household consumption term, as households spend a share ϕ of this new slice of income. Higher demand further increases output, and the process continues. The overall effect on output and income is

$$\Delta y = \Delta g + \phi \Delta g + \phi^2 \Delta g + \cdots = \frac{\Delta g}{1 - \phi}.$$

The fiscal multipliers increases with propensity to consume ϕ. The red curve in figure 8.1 depicts graphically the fiscal multiplier superior to one.

Criticisms:

- Constant marginal propensity
- Lack of constraints on aggregate supply
- Endogenous effects on wages and labor market

- Impact of potentially higher taxes to finance public spending?
- Dynamics? Public debt? Effect on interest rates?

8.2. Equilibrium Model

8.2.1. Endowment Economy

The Keynesian cross model implicitly assumes that the economy disposes of enough production capacity to fully attend the increase in aggregate demand following an increase in public spending. The opposite is true in an endowment economy.

Consider two-period model with endowments and no physical capital. Households receive y_t consumption goods each period. You can interpret the endowment as home production or as the equilibrium result of households that do not value leisure. Obviously, the fiscal multiplier in the endowment economy is zero.

But how does the equilibrium form? In period one, the household budget constraint

$$c_1 = y_1 - \tau_1 + b_0 \tag{8.4}$$

and government budget constraint

$$b_0 = \tau_1 - g_1 \tag{8.5}$$

combine to form the market clearing condition $y_1 = c_1 + g_1$. Public budget constraint (8.5) implies that an increase in public spending announced in $t = 1$ requires an increase in taxation of similar magnitude, $\Delta\tau_1 = \Delta g_1$. In turn, higher taxes pushes down household consumption through (8.4), $\Delta c_1 = -\Delta\tau_1$. The overall effect on aggregate demand is zero: $\Delta c_1 + \Delta g_1 = 0$.

The same story holds in period zero, and thus the fiscal multiplier continues to be zero.

8.2.2. Basic Analytics with Elastic Labor Supply

Reference: Woodford (2011).

We keep the two period structure. Since the derivation below holds equally to each period, we drop subscripts to simplify notation.

Period utility function $u(c) + v(1 - n)$, where u and v are increasing, concave and differentiable. Intratemporal optimality:

$$v'(1-n) = wu'(c) \tag{8.6}$$

Production function $f(n)$, concave. No physical capital. Firm optimization yields

$$f'(n) = w. \tag{8.7}$$

In equilibrium, $y = c + g$. By (8.6) and (8.7), marginal utility of leisure equals marginal cost of reducing labor hours, in utility units:

$$v'(1-n) = f'(n)u'(y-g). \tag{8.8}$$

Let

$$h(y) = -v(1 - f^{-1}(y)) < 0$$

be a household's "disutility of leisure" when the economy produces y consumption goods (the minus sign reverses the utility of leisure intepretation). Differentiating h gives

$$h'(y) = \frac{v'(1 - f^{-1}(y))}{f'(f^{-1}(y))} > 0.$$

Larger output demands lower leisure and thus lower utility derived from it. Since v and f are both concave, $h'' > 0$: producing more increases the marginal utility cost of further increasing production.

We can re-express equilibrium condition (8.8) in terms of the marginal benefit and cost of producing one more consumption good (rather than working one more hour), from the household standpoint:

$$h'(y) = u'(y-g). \tag{8.9}$$

Interpretation of (8.9)?

Define the elasticities

$$\eta_u = -\frac{u''(c)}{u'(c)} > 0 \qquad \text{and} \qquad \eta_h = \frac{h''(c)}{h'(c)} > 0.$$

Differentiating (8.9) yields

$$\eta_h \Delta y = \eta_u(\Delta g - \Delta y)$$

which leads to the fiscal multiplier:

$$\frac{\Delta y}{\Delta g} = \frac{\eta_u}{\eta_u + \eta_h} \in (0, 1). \tag{8.10}$$

Why fiscal multiplier $\in (0, 1)$? Intuition? What is wrong with $\Delta y/\Delta g = 0$? What is wrong with $\Delta y/\Delta g = 1$?

8.2.3. Real Interest and Policy Timing

The analytics above hold individually to periods zero and one. We now consider timing effects. First case: Suppose that, in period zero, the government announces an increase in public spending in period one, $\Delta g_1 > 0$, and that it will finance additional purchases through higher period-one (lump-sum) taxes $\Delta \tau_1 > 0$. How does that announcement change period-zero equilibrium?

Households anticipate lower consumption in period one $\Delta c_1 < 0$. In the new equilbrium:

$$\begin{aligned} q_0 u'(c_0) &= \beta u'(c_1 + \Delta c_1) \\ &> \beta u'(c_1) \end{aligned} \tag{8.11}$$

In the new equilibrium in period zero, either: higher bond prices, lower consumption, or both. Households attempt to increase savings to smooth consumption over time. Do they?

Equilibrium in the goods market characterized by market clearing and labor supply optimality condition:

$$f(n_0) = c_0 + g_0 \tag{8.12}$$

$$h'(1 - n_0) = u'(c_0) \tag{8.13}$$

(I assume an interior solution to labor supply choice, and continue to use the same definition of h.) (8.12) establishes a positive relationship between consumption and labor hours. (8.13) establishes a negative relationship. Therefore, there is, at most, a single solution to both the system (8.12)-(8.13). In conclusion, equilibrium c_0 and n_0 are unchanged in the new equilibrium.

By (8.11), equilibrium requires $\Delta q_0 > 0$, and thus $\Delta r_0 < 0$. Intuitively, households increase their demand for public bonds in an attempt to smooth the decline in public consumption stemming for larger spending in period one. But the stock of public bonds supplied by the government remains unchanged, as one can verify by writing down the government's budget constraint. The bond market clears at a higher bond price, and lower interest rate.

In the endowment economy case, consumption in period one falls as much as the increase in public spending, as households have no margin to increase labor hours. As a result, the economic motive to increase savings in period zero is stronger, and real interest rates falls by a larger amount.

Second case: the government announces higher spending and taxes in period zero, $\Delta g_1 = \Delta \tau_1 > 0$. Fiscal policy unchanged in period one. Applying the same logic as before, $\Delta c_1 = 0$. From the Euler equation

$$(q_0 + \Delta q_0)u'(c_0 + \Delta c_0) = u'(c_1) = q_0 u'(c_0).$$

Since $\Delta c_0 < 0$, $\Delta q_0 < 0$. Note that higher equilibrium real interest rates do *not* arise as a result of larger public debt. Instead, they arise as a by-product of intertemporal substitution by households.

Exercises

Exercise 7.1. Following the text's example, consider again the period-zero effects of the announcement of higher public spending in period one. Suppose that the government decides to finance the higher speding through higher taxes in period zero, keeping taxes unchanged in period one. Through the lenses of the model, is it correct to assert that households would, in that case, demand less bonds and, hence, that the effect over bond prices would be dampened? Explain.

Exercise 7.2. Consider a version of the two-period model with physical capital and no labor supply. The production function is $f(k) = rk$, where $r > 0$ is a fixed parameter. For simplicity, suppose that physical capital fully depreciates from one period to the next ($\delta = 1$, in the notation of chapter 2). Taxes τ_t are lump-sum. Households' utility function is

$$u(c_0) + \beta u(c_1),$$

where u is increasing, differentiable and concave.

(a) Write the market clearing condition in the goods market in periods one and two.

(b) Write the Euler equation governing households' choice of net wealth at the end of period zero.

(c) At the beginning of period one, the government announces an increase in public spending $\Delta g_1 > 0$, fully funded by an increase in taxation. What is the change in aggregate output Δy_1?

(d) At the beginning of period zero, the government announces an increase in public spending $\Delta g_0 > 0$, fully funded by an increase in taxation. What is the change in aggregate output in period zero Δy_0? Use the three equilibrium conditions you have derived to compute the fiscal multiplier in period-one output $\Delta y_1 / \Delta g_0$. What is the intuition for a negative multiplier?

(e) At the beginning of period *zero*, the government announces an increase in *perio-one* spending $\Delta g_1 > 0$, fully funded by an increase in τ_1. Use the three equilibrium conditions you have derived to compute the fiscal multiplier in period-one output $\Delta y_1 / \Delta g_1$. What is the intuition for a multiplier in the $(0, 1)$ range?

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